Karlsruher Institut für Technologie

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Riemannian Geometry | **Summer Term 2011**

Exercise Sheet 1 | **13.04.2011**

Exercise 1

a) Let $\pi : X \to Y$ be a map from a topological space X into a set Y. Show that there exists precisely one topology on Y, such that π is continuous and the following holds: a map $f : Y \to Z$ (Z a topological space) is continuous if and only if $f \circ \pi$ is continuous.

b) Let $\mathbb{R}/\mathbb{Z}$ be the factor group of $(\mathbb{R}, +)$ with canonical projection $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, $\pi(a) = a + \mathbb{Z}$. We can view $\mathbb{R}/\mathbb{Z}$ as a topological space with the quotient topology induced by π (see a)). Can you define a metric on $\mathbb{R}/\mathbb{Z}$ which induces this topology?

Exercise 2

Define the Möbius strip $M := [0,1] \times \mathbb{R}/\sim$ to be the quotient space where the pairs $(0,t)$ and $(1,-t)$ are identified. Show that

a) M admits a smooth atlas $\mathcal{A}$ compatible with its (quotient) topology,

b) $(M, \mathcal{A})$ is not orientable.

Exercise 3

The two maps

$$\varphi_1 := \mathrm{id}_{\mathbb{R}} \quad \text{and} \quad \varphi_2 : \mathbb{R} \to \mathbb{R},\ x \mapsto x^3$$

define smooth structures $\mathcal{A}_1$ and $\mathcal{A}_2$ on the topological manifold $\mathbb{R}$. Show that $\mathcal{A}_1 \neq \mathcal{A}_2$ but $(\mathbb{R}, \mathcal{A}_1)$ and $(\mathbb{R}, \mathcal{A}_2)$ are diffeomorphic.

Exercise 4

Let $\mathbb{P}^n(\mathbb{R})$ be the n dimensional real projective space. Show that

$$\{(U_i, \varphi_i) \ : \ i = 1, \dots, n+1\}$$

with

$$U_i := \{[(x_1, \dots, x_{n+1})] \ : \ x_i \neq 0\}$$

and charts

$$\varphi_i : U_i \to \mathbb{R}^n, [(x_1, \dots, x_{n+1})] \mapsto \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right)$$

is a smooth atlas on $\mathbb{P}^n(\mathbb{R})$.
