Karlsruher Institut für Technologie

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Riemannian Geometry — Summer Term 2011

Exercise Sheet 03 — **27.04.2011**

Exercise 1

a) Show that the special linear group $\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) = 1\}$ is a submanifold of $\mathbb{R}^{n\times n}$.

b) Let

$$\begin{aligned} \mu : \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R}) &\to \mathrm{SL}_n(\mathbb{R}) \\ (A, B) &\mapsto A \cdot B \end{aligned}$$

be the group multiplication of $\mathrm{SL}_n(\mathbb{R})$ and

$$\begin{aligned} \iota : \mathrm{SL}_n(\mathbb{R}) &\to \mathrm{SL}_n(\mathbb{R}) \\ A &\mapsto A^{-1} \end{aligned}$$

be the inverse map. Show that μ and ι are differentiable maps.

c) Show that

$$\{X \in \mathbb{R}^{n\times n} \mid \mathrm{tr}(X) = 0\}$$

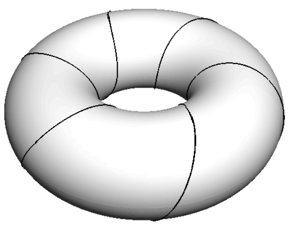
is the tangent space $T_I\mathrm{SL}_n(\mathbb{R})$ at the identity matrix I.

Exercise 2

Let $T := \mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$ be the two torus with its natural smooth structure, $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ the canonical projection and $\gamma(t) := \{(ta, tb) \mid t \in \mathbb{R}\} \subset \mathbb{R}^2$ for $a, b \in \mathbb{R}$. Show that

a) $\pi \circ \gamma$ is an immersion.

b) If $b = 0$ or $\frac{a}{b} \in \mathbb{Q}$ then $\pi \circ \gamma$ is a submanifold of T diffeomorphic to S^1.



c) If $\frac{a}{b} \notin \mathbb{Q}$ then $\pi \circ \gamma$ is a dense subset of T.
Hint: Use the following fact from the geometry of numbers: Let $x \in \mathbb{R} - \mathbb{Q}$. Then there exists infinitely many integers $p, q \in \mathbb{Z}$ such that $|x - \frac{p}{q}| < \frac{1}{q^2}$. You may find this fact in the book of Cassels, John W. S.; An Introduction to the geometry of numbers.

Exercise 3

Let $S^3 \subset \mathbb{R}^4$ be the three dimensional sphere. Show that there exists three vectorfields $X_1, X_2, X_3 : S^3 \to TS^3$ such that $\{X_1(x), X_2(x), X_3(x)\}$ is a basis of T_xS^3 for all $x \in S^3$.