Karlsruher Institut für Technologie

Institute for Algebra and Geometry
HDoz.Dr. Oliver Baues
Dipl.Math. Johannes Riesterer

Riemannian Geometry **Summer Term 2011**

Holiday Sheet **12.07.2011**

Exercise 1

A Riemannian manifold (M, g) is called a homogeneous space, if for all $x, y \in M$ there exists $\psi \in \operatorname{Isom}(M, g)$, such that $\psi(x) = y$. Show that a homogeneous space is complete.

Exercise 2

Let (M, g) be a connected Riemannian manifold and $\phi, \psi \in \operatorname{Isom}(M, g)$ be isometries. Show that, if $d\phi_p = d\psi_p$ and $\phi(p) = \psi(p)$ for one $p \in M$, then $\phi = \psi$.
Hint: Show that the set $\{q \in M \mid \phi(q) = \psi(q)\}$ is an open and closed subset of M.

Exercise 3

Let G be a Lie group, and let h a bi-invariant metric (Compare Exercise Sheet 11 Exercise 3). Moreover let D be the Levi-Civita connection for h. Let $\mathfrak{g}$ be the vector space of left-invariant vector fields on G.

a) Show that $D_X Y = \frac{1}{2}[X, Y]$ and $R(X, Y)Z = \frac{1}{4}[[X, Y]Z]$ holds for all left-invariant vector fields $X, Y \in \mathfrak{g}$.

c) Let $X_e, Y_e \in T_e G$ be orthogonal with respect to h. Show that for the sectional curvature at the point $e \in G$

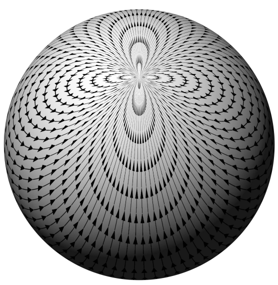
$$K_e(X_e, Y_e) = \frac{1}{4}||[X, Y](e)||_h$$

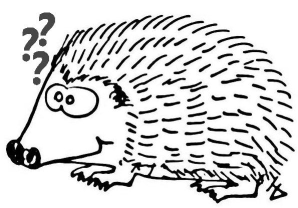
holds, where X and Y are the left invariant vector fields $X(e) = X_e$ and $Y(e) = Y_e$. Conclude that the sectional curvature is non-negative everywhere.

d) A Riemannian manifold is called flat, if its sectional curvatures are zero everywhere. Characterise all Lie groups with flat bi-invariant metric.

Exercise 4

Let S^2 be the two sphere. Show that, for every vector field $X : S^2 \to TS^2$, there exists $p \in M$ with $X(p) = 0$.
Hint: See S.Gallot, D.Hulin, J.Lafontaine; *Riemannian Geometry*, Theorem 1.41 or John W. Milnor; *Topology from the Differential Viewpoint*, page 31.





We wish all participants of the lecture happy holidays and all the best for their further studies.