Deformations and rigidity of lattices in solvable Lie groups

joint work with Benjamin Klopsch

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- Rigidity of lattices in Lie groups
  - Rigidity theorem of Mal'cev and Saitô.
- Rigid embedding into algebraic groups
- Quantitative description of the rigidity problem
- The "zoo" of solvable Lie groups
- Unipotently connected groups
- Finiteness theorem for $\mathcal{D}(\Gamma, G)$
- Strong rigidity and structure set
- Representation of the structure set

Let G be a (connected) Lie group, $\Gamma \leq G$ a discrete subgroup.

Definition

Γ is called a *lattice* in G if G/Γ is compact (or has finite volume).

Example

$\mathbb{Z}^n \leq \mathbb{R}^n$, $\mathrm{SL}(n, \mathbb{Z}) \leq \mathrm{SL}(n, \mathbb{R})$.

Γ is a discrete approximation of G.

Q: How closely related are Γ and G?

Mostow strong rigidity

Theorem

Let G and G' be semisimple Lie groups of non-compact type with trivial center, not locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$*, and,* $\Gamma \leq G$*,* $\Gamma' \leq G'$ *irreducible lattices. Then every isomorphism*

$$\varphi : \Gamma \to \Gamma'$$

extends uniquely to an isomorphism of ambient Lie groups

$$\hat{\varphi} : G \longrightarrow G' \, .$$

$\Gamma \leq G$ a lattice.

Definition

Γ is *rigid* if for any isomorphism $\varphi : \Gamma \to \Gamma'$, where $\Gamma' \leq G$ is a lattice, there exists an extension $\hat{\varphi} : G \to G$.

Definition

Γ is *weakly rigid* if for any automorphism $\varphi : \Gamma \to \Gamma$ there exists an extension $\hat{\varphi} : G \to G$.

Important examples in the context of solvable Lie groups:
Auslander 1960, Milovanov 1973, Starkov 1994

Rigidity theorem of Mal'cev (1949) and Saitô (1957)

Theorem

Let G and G' be simply connected nilpotent solvable Lie groups of real type and $\Gamma \leq G$, $\Gamma' \leq G'$ lattices. Then every isomorphism

$$\varphi : \Gamma \to \Gamma'$$

extends uniquely to an isomorphism of ambient Lie groups

$$\hat{\varphi} : G \longrightarrow G' .$$

In particular, Γ a lattice in a simply connected solvable Lie group of real type. Then Γ is rigid in G.

Rigid embedding into algebraic groups, Mostow 1970

Let Γ be polycyclic, torsionfree.

Theorem (Existence)

There exist a $\mathbb{Q}$-defined linear algebraic group $\mathbf{A}$ and an embedding $\iota\colon \Gamma \hookrightarrow \mathbf{A}$ such that $\iota(\Gamma) \subseteq \mathbf{A}_{\mathbb{Q}}$ and

(i) $\iota(\Gamma)$ *is Zariski-dense in* $\mathbf{A}$,

(ii) $\mathbf{A}$ *has a strong unipotent radical, i.e.* $\mathrm{C}_{\mathbf{A}}(\mathrm{Rad_u}(\mathbf{A})) \subseteq \mathrm{Rad_u}(\mathbf{A})$,

(iii) $\dim \mathrm{Rad_u}(\mathbf{A}) = \mathrm{rk}\, \Gamma$.

The group $\mathbf{A} = \mathbf{A}(\Gamma)$ is called an *algebraic hull* for Γ.

Proposition (Rigidity of the algebraic hull)

Let $\Gamma \leq \mathbf{A} = \mathbf{A}(\Gamma)$, *and* $\Gamma' \leq \mathbf{B} = \mathbf{A}(\Gamma')$ *be* $\mathbb{Q}$*-defined algebraic hulls. Then every isomorphism*

$$\varphi : \Gamma \to \Gamma'$$

extends uniquely to a $\mathbb{Q}$*-defined isomorphism of algebraic groups*

$$\Phi \colon \mathbf{A} \to \mathbf{B}.$$

The space of lattice embeddings

$$\mathcal{X}(\Gamma, G) := \{\varphi\colon \Gamma \hookrightarrow G \mid \varphi(\Gamma) \text{ is a lattice in } G\}$$

The *deformation space* of Γ is

$$\mathcal{D}(\Gamma, G) = \mathrm{Aut}(G) \backslash \mathcal{X}(\Gamma, G) \, .$$

Definition

Γ is called *deformation rigid* if

$$\mathcal{D}(\Gamma, G)_0 = \mathrm{Aut}(G)_0 \backslash \mathcal{X}(\Gamma, G)_0 = \{*\} \, .$$

Examples:
- $\mathbb{Z}^3$ is *not* deformation rigid in $\widetilde{E(2)}$.
- Milovanov 1973: non-deformation rigid Γ in G of type (E), $\dim G = 5$.

Auslander 1973/Starkov 1994: Classification via the eigenvalues λ of the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$.

- *nilpotent* N
  $(\mathbb{R}^n, +)$, Heisenberg-group $H_3(\mathbb{R})$.
- *real type* R
  3-dimensional unimodular group Sol.
- *exponential type* E no λ on the unit circle, except 1
- *type* A all $\mathrm{Ad}(g)$ are either unipotent, or have a λ with $|\lambda| \neq 1$
  First example by Auslander 1960, $\dim G = 5$.
- *type* I (*imaginary type*) all λ on the unit circle
- mixed

$$N \subset R \subset E \subset A\,,\ I \cap A = N$$

Definition (F. Grunewald, D. Segal[1])

Let $G \leq \mathrm{GL}(N, \mathbb{R})$ be a solvable Lie subgroup. Then G is called *unipotently connected* if $G \cap \mathrm{u}(G)$ is connected.

We say G is unipotently connected if it is unipotently connected as a subgroup of its algebraic hull A_G.

Proposition

Every Γ has a finite index subgroup Γ' which is a Zariski-dense lattice in a unipotently connected group G'.

The folowing are equivalent:

1. G is unipotently connected.
2. G is of type A in the A-S classification.

[1] On affine crystallographic groups, JDG **40**, 1994

Theorem (A)

Let G be simply connected, and unipotently connected. Then, for every Zariski-dense lattice Γ of G, the deformation space $\mathcal{D}(\Gamma, G)$ is finite.

Both assumptions (u-connected) and Zariski-dense are necessary.

Example

There exists a pair (G, Γ), G is of mixed type, $\Gamma \leq G$ is Zariski-dense, $\dim G = 12$, $\dim N(G) = 8$, $\operatorname{rk} \operatorname{Fitt}(\Gamma) = 10$, such that $\mathcal{D}(\Gamma, G)$ is countably infinite.

Let Γ be a lattice in a simply connected, solvable Lie group G.

Corollary (1)

There exists a finite index subgroup Γ' of Γ which embeds as a Zariski-dense lattice into G' such that the deformation space $\mathcal{D}(\Gamma', G')$ is finite.

Corollary (2)

If G is unipotently connected, then there exists a finite index subgroup $\mathrm{Aut}^\circ(\Gamma)$ of $\mathrm{Aut}(\Gamma)$ such that every element of $\mathrm{Aut}^\circ(\Gamma)$ extends to an automorphism of G.

Indeed, in Corollary (2) one may take

$$\mathrm{Aut}^\circ(\Gamma) = \mathrm{C}_{\mathrm{Aut}(\Gamma)}(\Gamma / \mathrm{Fitt}(\Gamma))$$

.

Let $\Gamma \leq G$ be a Zariski-dense lattice.

Definition

Γ is called *strongly rigid* if every isomorphism $\varphi : \Gamma \to \Gamma'$ where Γ' is a Zariski-dense lattice in some G' extends to an isomorphism

$$\hat{\varphi} \colon G \to G'.$$

Define the *structure set* for Zariski-dense embeddings of Γ as

$$\mathcal{S}^{\mathsf{Z}}(\Gamma) = \{\varphi : \Gamma \hookrightarrow G' \mid \varphi(\Gamma) \text{ is a Zariski-dense lattice in } G'\}/\sim$$

Theorem (B)

The structure set $\mathcal{S}^{\mathsf{Z}}(\Gamma)$ is either countably infinite or it consists of a single element. The structure set consists of a single element if and only if Γ is a lattice in a solvable Lie group of real type.

Corollary (3)

Let G be unipotently connected. Then Γ is either strongly rigid or there exist countably infinite pairwise non-isomorphic simply connected (and also unipotently connected) solvable Lie groups which contain Γ as a Zariski-dense lattice.

$$\mathcal{G}(\Gamma) = \{ G \leq \mathbf{A}(\Gamma)_{\mathbb{R}} \mid G \text{ simply connected, solvable Lie subgroup and } \Gamma \text{ a (Zariski-dense) lattice in } G \}$$

For every $\varphi : \Gamma \to G$, exists a unique extension $\Phi : \mathbf{A}_\Gamma \to \mathbf{A}_G$.

Proposition

The structure map

$$\epsilon : \mathcal{S}^{\mathsf{Z}}(\Gamma) \longrightarrow \mathcal{G}(\Gamma) , \quad [\varphi]_{\mathcal{S}^{\mathsf{Z}}(\Gamma)} \mapsto \Phi^{-1}(G) .$$

is a bijection.