Proof: First observe from the BT fixed pt lemma (see I.14) that a finite group $\Gamma$ which acts on a CAT(0) space $X$ has a non-empty, convex set of fixed points $Fix(\Gamma) \subset X$.

In addition one can show that cocompact actions only have finitely many isotropy subgroups, i.e. \( \Gamma_x = \{ g \in \Gamma \mid g.x = x \} \). (B+$\pm$.8.5 (a))

This implies the statement.

Next time: More properties of CAT(0) groups & boundaries

End of 4th lecture

### III.12 Properties of CAT(0) groups

Let $G$ be a CAT(0) group, then

1. $G$ is finitely presented.

2. There exists a finite model for the universal space of proper $G$-actions $EG = \text{proper } G$-CW-complex $E_G$ such that $E_G^H$ is contractible for finite $H \leq G$.

(Weak)

3. Every solvable subgroup is virtually $\mathbb{Z}^n$.

4. Direct product with a CAT(0) group is CAT(0)

4. $G$ is hyperbolic
(5) Suppose $G \cong \mathbb{R} \times$ isom., proper, cocompact, $X_{\text{CAT}(0)}$. Then $G$ is hyperbolic if and only if $X$ does not contain an isometrically embedded copy of a Euclidean plane.

(6) The word and conjugacy problem are solvable for $G$. (open: isomorphism problem)

References: See Lück: "Survey on geometric group theory", and the references given in Chapter 8.

Rmk: to (5): $\mathbb{Z}^2 \cong \mathbb{R}^2$ geometrically and is not hyperbolic.

So this is "the only" obstruction against being $X_{\text{CAT}(0)}$.

To (6): The word problem:

Given $w \in G$, decide whether it represents a word in the generators of $G$.

Conjugacy problem: decide whether two given words are conjugate.

more to (5): see next theorem.
Let $G$ be finitely generated. Then $G$ acts properly cocompactly by isometries on $E^n$. $G$ contains a finite index subgroup $A \cong \mathbb{Z}^n$.

Compare with:

Suppose $G \cong \text{CAT}(0)$ and $\exists$ subgroup $A \cong \mathbb{Z}^n$ in $G$, then $\exists$ an isom. embedding $F : E^n \to X$ such $F(E^n)$ is stabilized by $A$.

In this case, $A$ has a torus action on $F(E^n)$.

We close section III with the following:

### III.15 Non-trivial Non-examples

1. Gesztesy (1994) "the autom. grp of a free grp is not Cat(0)" $G = \langle a, b, c, t | tat^{-1} = a, tbt^{-1} = b, tc \gamma^t = c \gamma^t \rangle = \langle \gamma^{-1} | tat^{-1} = a, b^{-1}tb = at, c^{-1}tc = a^2t \rangle$

$\sim a, t$ commute and generate $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset G$.

$\text{Supp. } G \cong \text{CAT}(0)$ $\Rightarrow \exists$ isom. embedded copy of $\mathbb{T}^2$ in $X$ on which $a, t$ act by translations.

b.w.
By Prop. III.3 (2) we have
\[ |g| = |ghg^{-1}| \] for any \( g, h \in G \), \( |\cdot| \) = translation length

\[ \Rightarrow |t| = |bc^{-1} t c| = |a^2 t| \]

Hence \( G \) (and any group having \( G \) as a subgroup)
can not be CAT(0).

Gershen shows: \( \text{Out}(F_n) \) for \( n \geq 3 \)
and \( \text{Aut}(F_2) \) are not CAT(0).

b) \((BH: II.7.26)\)

Mapping class group of a surface of genus \( g \geq 3 \) is not CAT(0).

eg. \( \text{IICG}(T^2) \cong \text{SL}_2(\mathbb{Z}) \)

\( \mathbb{Z}^2 \)

Rmk III.16

\( \text{Out}(F_n) \) acts properly discontinuously on
Cullers-Vogtmann outer space \( CV_n \), the
analog of a Teichmüller space of a hyperbolic
surface.
IV. Boundaries

We first come back to (CAT(0)) spaces and their (individual) isometries. One (of many) motivations to study boundaries is that the parabolic isometries can not be studied using Fix(.) and Min(.) ( = ∅).

IV.1 Def. (asymptotic rays)

Let \( y, y' \) be two geodesic rays. We say \( y \) and \( y' \) are asymptotic if there const. \( c > 0 \) s.th. \( \forall t \geq 0 \)

\[ d(y(t), y'(t)) \leq c. \]

Write \( y \sim y' \) and \([y]_n \) for the equivalence class.

The (ideal) boundary of \( X \) is

\[ \partial X := \{ [y]_n | y \text{ a good ray in } X \} . \]

Remark: Same definition makes sense for \( \delta \)-hyperbolic spaces.

Examples:

\[ \mathbb{H}^2 \]
Prop.

Let \( X \) be a complete metric space. For any \( p \in X \) and any \( x \in X \), there exists a unique geodesic ray \( \gamma : [0, \infty) \to X \) such that \( \gamma(0) = x \) and \( \gamma([0, n]) = p \).

Sketch of proof:

- Uniqueness can be deduced from the convexity property of the distance function.

\[ d(c(t), c'(t)) \text{ is bounded, non-negative, convex}, \quad d(c(0), c(0)) = 0 \]
\[ \Rightarrow d(c(...)) = 0 \]

- Existence:

Given \( \gamma \), given \( \gamma : [0, \infty) \to X \) a geod. ray s.t., \( \gamma([0, n]) = p \). Put \( y = \gamma(y) \).

Show: \( \gamma(t) \) converges, for a fixed \( s \)

Define \( \gamma(s) = \lim_{t \to 0} \gamma(t) \)

and compute that \( \gamma([0, n]) = p = \gamma(y) \).

Details see BH II.8.2/8.3.
This allows us to consider the following:

**IV.3 Def.** (Local representatives of \( \partial X \))

Fix (any) base point \( x_0 \in X \) and define

\[
\partial_{x_0} X = \{ y \mid y \text{ geod. ray, } y(0) = x_0 \}
\]

Remark \( \partial_{x_0} X \) is in canon. bijection with \( \partial X \) (loc. of IV.2) and \( \partial_y X \) \( \forall y \in X \).

We use these local representatives to define a topology on \( \partial X \):

**IV.4 Def.** Basis for the visual topology

\( y \in \partial_{x_0} X, \varepsilon, R > 0 \) s.t. \( R \gg 1 \) and \( \varepsilon \ll 1 \).

Define

\[
N(y, \varepsilon, R) = \{ y' \mid d(y(R), y'(R)) < \varepsilon \}
\]

and let \( \tau_{x_0} \) be the topology generated by the valid basis

\[
\{ N(y, \varepsilon, R) \mid y \in \partial_{x_0} X, \varepsilon, R > 0 \}
\]

This is called the **visual topology** \( \tau_{x_0} \) on \( \partial X \). Denote the top. space by \( \partial_{x_0} X \).

**Fact:** \( \partial_{x_0} X \) homeom. to \( \partial_y X \) \( \forall y \neq x_0 \) and hence the top. on \( \partial X \) is uniquely defined.
4.5 Examples: False plane, \( \partial_\infty X = S^{d-1} \)

- \( X = \mathbb{E}^d \), \( \partial_\infty X = S^{d-1} \)
- \( X = \mathbb{H}^d \), \( \partial_\infty X = S^{d-1} \)
- \( X = \mathbb{T}_d \), \( \partial_\infty X = \mathbb{C} \) Cantor set

Facts:
1) \( \partial_\infty (X \times Y) = \partial_\infty X \ast \partial_\infty Y \)
   \[ \uparrow \uparrow \]
   CAT(0) spaces
   In particular, \( \partial_\infty (X \times \mathbb{R}) = \Sigma \partial_\infty X \)

2) (Ruan) \( g \) hyperbolic isom. on a CAT(0) space \( X \), then \( \partial_\infty \text{Fix}(g) = \text{Fix}(\bar{g}) \)
   where \( \bar{g} : \partial_\infty X \to \partial_\infty X \) the induced homeomorphism of the boundary.

3) \( X \) proper CAT(0), \( f \) parabolic isom. on \( X \),
   then \( \exists \ p \in \partial_\infty X \) fixed by \( f \)
   s.t. \( f \) leaves every horosphere centered at \( p \) invariant.
   (B.H., Fujiwara-Shiroya-Yamagata 03)

4) \( \partial_\infty X \) is not invariant under quasi-isometry and hence not uniquely determined by an isometry gap of \( X \).
   See next example.
Take $A_T$ to be the RAAG defined by the graph $I = \{a, b, c, d\}$. We have $A_T = \langle a, b, c, d \mid [a, b] = [b, c] = [c, d] = 1 \rangle$.

Then $A_T$ acts geometrically on its Salvetti-Komplex $X = S_T$.

The $S_T$ is made out of $3$-gons:

Now vary the angle between the curve $c$ & $b$ in the middle torus:

This yields a new metric on the resulting space $X_x$ for any acute $x \leq \frac{\pi}{2}$.

CK shows: $d_{x_1}^x$ is not homeo to $d_{x_2}^x$. 
In fact, Wilson ("05) showed:

\[ \partial x \neq \partial x' \quad \forall \; 0 < \alpha < \beta \leq \frac{\pi}{2} \]

IV.8 Some remarks:

1) One can put a different natural topology on \(\partial X\), the **Tits topology**, which is induced by a metric on \(\partial X\), the **Tits metric**. Denote space by \(\partial X\).

   It is defined by measuring angles of geodesic rays at \(x_0 \in X\).

2) The **Tits metric** allows a more refined study of (induced) maps \(\partial X \to X\).

3) Compactifications \(\overline{X} = X \cup \partial X\) or \(X \cup \partial X\)

   are used in various places to e.g. study the dynamics of group actions at \(\infty\).

4) Chauney-Sultan defined a new boundary for CAT(0) spaces, the **contracting boundary** (or Morse boundary) which is q.i. invariant and somehow captures the "hyperbolic part" of a CAT(0) space.

   Very little is known about those.