3. Bruhat–Tits Fixed Point Theorem

History:
Elie Cartan (1928): simply conn., complete Riem.
mfd $\mathbf{M}$ of non-pos. sect. curvature
$\Rightarrow$ every finite set has a center
i.e. the map $x \mapsto \frac{1}{n} \sum_{i=1}^{n} d(x, x_i)^2$ for a fixed
$n$-tuple $x_1, \ldots, x_n$ has a minimum.

As a consequence he obtains that compact isometries $g$ of $\mathbf{M}$
always have a fixed pt.

François Bruhat – Jacques Tits (1972):
same is true for Euclidean buildings (which
are a subclass of CAT(0) spaces).

Def. The **circumradius** (short: radius) of a bounded
set $Y$ in a metric space $(X, d)$ is given by
\[
 r(Y) := \inf_{x \in X} \left( \sup_{a \in A} d(x, a) \right) \\
= \inf_{x \in \mathbb{R}^+} \left\{ y \in \mathbb{R} : y \leq B(x), \ x \in X \right\}
\]

I.13 Existence of a center (Bruhat–Tits/Brown)

$X$ a complete CAT(0) space and $Y$ a bounded
subset of $X$, then there exists a unique $p \in X$

$s.t. \quad B_{x}(y)(p) \geq Y$.

We call $p$ the center of $Y$. 
For the proof of I.13 we need the following formula in \((\mathbb{R}^2, \text{de}^2)\):

\[ P_t := (1-t)x + ty \in x\bar{y} \text{ for } t \in [0,1] \]

Then \(\forall z \in \mathbb{R}^2\)

\[
\begin{align*}
  d^2(z, P_t) &= (1-t) d^2(z, x) + td^2(z, y) \\
               &= t(1-t) d^2(x, y)
\end{align*}
\]

Proof of I.13: We first prove uniqueness:

For a bounded set \(Y\) put \(\tau(x, y) := \sup \{d(x, y) : y \in Y\}\).

i.e. \(\tau(x, y)\) is the smallest number such that \(y \in \overline{B}_\tau(x)\).

For any pair \(a, b\) of points in \(X\) consider an arbitrary \(z \in Y\) the triple \((a, b, z)\) and its comparison triangle in \((\mathbb{R}^2, \text{de}^2)\):

Then

\[
\begin{align*}
  d^2(z, m) &\leq d^2(z, \overline{a}) = \frac{1}{2} d_e^2(z, \overline{a}) + \frac{1}{2} d_e^2(z, \overline{b}) - \frac{1}{4} d_e^2(a, b) \\
  &= 4 d^2(z, m) \leq 2d^2(z, a) + 2d^2(z, b) - d^2(a, b) \\
  &= 4 \tau^2(m, Y) \leq 2 \tau^2(a, Y) + 2 \tau^2(b, Y) - d^2(a, b)
\end{align*}
\]

as \(z \in Y\)

by construction of \(\tau(Y)\)
If we assume that there exist two circumcenters \( q, q' \) in \( X \) we may put \( a := q \) and \( b := q' \) in (**) .

This implies
\[
0 \leq d^2(a, b) \leq 4r^2(Y) = 4r^2(Y) = 0,
\]

Hence \( q = q' \).

To show existence:
Take a sequence \( (x_n) \) of pts in \( X \) such \( r(x_n, y) \to r(y) \).

Apply (**) to \( a := x_n \) \( b := x_m \) for some \( m, n \).

The right hand side then can be made arbitrarily small by making \( m, n \) large \( \Rightarrow (x_n) \) is Cauchy

and has a limit \( x \in X \). (as \( X \) is complete).

One can check \( r(x, y) = r(y) \).

From I.13 we will be able to deduce

I.14 **Brouwer Fixed Point Theorem**

If \( G \subseteq \text{Isom}(X) \), \( X \) a complete CAT(0) space.

If \( G \) stabilizes a bounded subset in \( X \)
(e.g. if \( G \) is finite) then \( X^G \neq \emptyset \) and convex.

\[
\text{put } X^G := \{ x \in X \mid gx = x \text{ for all } g \} \]
proof of 1.14: The fact that $X^G = \emptyset$ follows immediately from 1.14 because the circumcenter is fixed by $G_1$ of the fixed bounded set (e.g. a finite orbit).

To see that it is convex is left as an exercise (hint: use the fact that $X$ is uniquely geodesic).

Examples: buildings, symmetric spaces.
4 Symmetric spaces

I.15 Def. A symmetric space is a Riemannian manifold \( M \) such that the geodesic symmetry at \( x \) is a global isometry \( A_x M \). Equivalently, for \( x \in M \), the differential equals \( -\text{id} \) on \( T_x M \), and \( \xi_x(x) = x \).

I.16 Ex. \( S^n \), \( E^n \), \( H^n \)

We say that a symmetric space \( M \) is of non-compact type if it has non-positively sectional curvature (and no non-trivial Euclidean factor).

I.17 Fact: the connected component of \( \text{id} \) in \( \text{Isom}(M) \) of a symmetric space \( M \) of non-compact type is a semi-simple Lie group \( G \) with trivial center and no compact factors.

I.18 On the contrary, from every even Lie group one can construct a symmetric space \( M \) of non-compact type as follows:

\[ M = G / K \]

\( M \) is a compact space, where \( K \) is a maximal compact subgroup of \( G \).
Example

Take $G = \text{SL}(\mathbb{R}) \quad K = \text{SO}(n)$ - orthogonal. $G^T G = I$

$G/K = \mathbb{R}$ can be identified with the collection of scalar products on $\mathbb{R}^n$ for which the unit balls have the same volume as the one for $d_{eucl}$.

Define a distance function on $\mathbb{R}$ as follows:

$x_i := (\ldots); \text{ scalar products } i = 1, 2 \ldots \nu = \text{basis of } \mathbb{R}^n$ s.t. they correspond to $(\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_n)$

put $d(x_1, x_2) := \sqrt{\sum_{i=1}^{\nu} \left( \log \frac{\lambda_i}{\mu_i} \right)^2}$

Nontrivial fact: $(\mathbb{R}, d)$ is a CAT(0) space.

Properties

1. every symm. sp. of non-compact type contains a subset isometric to some $(\mathbb{R}^n, d_{eucl})$. The maximal such $n$ is called its rank.

2. $Ax_1, x_2 \in \mathbb{R}$: symm. sp. of non-compact type and the line there exists a $k$-flat containing $x_1$ and $x_2$.

3. the flats from (2) are equipped with a (natural) action of an infinite (euclidean) reflection group.

4. dimension of the symm. space is larger than the dim. of its max. flats (i.e., its rank).
5 Euclidean buildings

The building blocks of buildings are:

I.21 "Def." Euclidean Coxeter complexes

$W \leq \text{Isom}(\mathbb{R}^n)$ discrete reflection group

ie. disc. subgp generated by orthog. refl. along hyperplanes in $\mathbb{R}^n$

$\triangleright$ collection of refl. hyperplanes $H$ in $\mathbb{R}^n$ is locally finite

$\triangleright$ it determines a cellular decomposition of $\mathbb{R}^n$ the underlying simp. struc. of which we call Euclidean Coxeter Cplx

List of all Eu. Cox.Cplx. in dim 2:

$\hat{A}_2, \hat{C}_2 = \hat{C}_2$.

See above

in dim 1: $\hat{A}_1$

I.22 "Def." Euclidean building

A Euclidean building is a simp. Cplx $X$ which satisfies $(B1)$: Any two simplices are contained in a common supl. Cplx (called apartment) that is isom. to some Euclidean Cox. Cplx.

Combinatorially, i.e. as simp. Cplx.

$(B2)$: Given $A, A'$ two apartments in $X$ there exists an isom. $A \to A'$ fixing $A, A'$ pointwise.
Examples / Characterizations / Facts:

1) The 1-dim Euclid. Edges are exactly the (simple) trees w/o leaves. (Diagram)

2) Higher dimensional Euclid. Edges arise (for example) from $\text{SL}_n(\mathbb{Q}_p)$ + valuation. And $\text{SL}_n(\mathbb{Q}_p)$ acts transitively on max. simplices.

3) Every Eucl. Edge has a metric realization $(I_x, \bar{d})$ s.t. $I$ agents $A \subseteq X$, the rest of $d$ to $1/4$ is the Euclidean metric. The space $(I_x, \bar{d})$ is a CAT(0) space, (complete)

4) (Kleiner) $X$ a locally compact CAT(0) space of geometric dimension $n$. If any pair of points is in a common $n$-flat, then $X$ is the metric realization of a Eucl. Edge.

$\text{geom.dim} := \sup \text{ over all compact subsets } K \subseteq X \text{ of the topol. dim. of } K$

See also Kleines: "The local structure of length-spaces with curv. bdd. above" (1995) for alternative descriptions.

End of 2nd lecture

[AB]: Abramenko - Brown: Buildings - Theory and Applications