III CAT(0) groups and Isometries of CAT(0) spaces

Isometries of a metric space X fall into three categories, which we will now introduce.

III 1 Def. (X,d) metr. space, $\gamma \in \mathrm{Isom}(X)$ = isometries of X

then the <u>displacement function</u>
$d_\gamma : X \to \mathbb{R}_+$ of γ is defined by $d_\gamma(x) := d(\gamma.x, x)$
and the <u>translation length</u> of γ is
$$|\gamma| := \inf\{ d_\gamma(x) \mid x \in X\}.$$
The <u>Min-set</u> of γ is $\mathrm{Min}(\gamma) := \{x \in X \mid d_\gamma(x) = |\gamma|\}$.
Put <u>$\mathrm{Min}(\Gamma)$</u> $:= \bigcap_{\gamma \in \Gamma} \mathrm{Min}(\gamma)$ for any subgrp $\Gamma \leq \mathrm{Isom}(X)$.

Ex. Consider $f_\lambda : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto x + \lambda$ for some $\lambda \in \mathbb{R}^n$.

Then $|f_\lambda| = \|\lambda\|$ and $\mathrm{Min}(f_\lambda) = \mathbb{R}^n$.

We can use the displ. fct. and the min-set to separate the isometries of X into 3 classes:

Def. III.2 (3 types of isometries)

An isometry γ of a metr. sp. X is called
(i) <u>elliptic</u> if $|\gamma| = 0$ ($\Leftrightarrow \gamma$ has a fixed pt)
(ii) <u>hyperbolic</u> (or axial) if d_γ has a strictly positive minimum
(iii) <u>parabolic</u> if $\mathrm{Min}(\gamma) = \emptyset$ ($\Leftrightarrow d_\gamma$ does not attain a minimum)

In addition, we say γ is <u>semi-simple</u> if $\mathrm{Min}(\gamma) \neq \emptyset$.

Prop. III.3 (Properties of min-sets)

(X,d) a metr. space, $\gamma \in \mathrm{Isom}(X)$, $\Gamma \leq \mathrm{Isom}(X)$.

Then (1) $\mathrm{Min}(\gamma)$ is γ-invariant, and $\mathrm{Min}(\Gamma)$ is Γ-invariant.

(2) $\alpha \in \mathrm{Isom}(X)$, then $|\gamma| = |\alpha\gamma\alpha^{-1}|$ and $\mathrm{Min}(\alpha\gamma\alpha^{-1}) = \alpha . \mathrm{Min}(\gamma)$.

(3) X CAT(0) $\Rightarrow$ d_γ is convex and $\mathrm{Min}(\gamma)$ is a closed, convex set.

(4) If $C \subset X$ is non-empty, complete, convex and $\gamma . C = C$, then

- $|\gamma| = |\gamma|_C|$ and
- γ semi-simple $\Leftrightarrow$ $\gamma|_C$ semi-simple
- $\mathrm{Min}(\gamma) \neq \emptyset \Leftrightarrow C \cap \mathrm{Min}(\gamma) \neq \emptyset$

Proof:

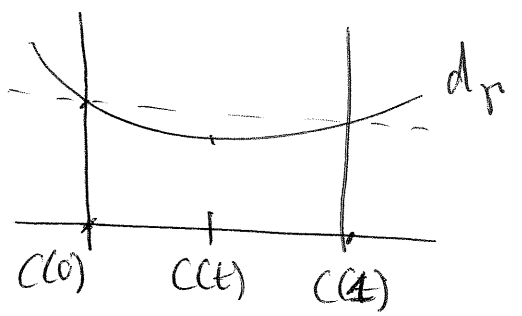
(1) and (2) follow easily from the definition.

(3) In Le II.2 we have seen that d is a convex metric. (*)

For d_γ to be convex we need to check that for any geodesic $\underset{t \in [0,1]}{c(t)}$ we have

$$d_\gamma(c(t)) \leq (1-t)\, d_\gamma(c(0)) + t \cdot d_\gamma(c(1)), \ \forall t,$$

Hence the statement.



(*) $d(c_1(t), c_2(t))$
$\leq (1-t)\, d(c_1(0), c_2(0))$
$+ t \cdot d(c_1(1), c_2(1))$.

Put $c_1 = c$, $c_2 \equiv$ constant geodesic

(4) To see this item consider $p: X \to C$ projection onto convex set.
One can show (use BH II.2.4 p.176) that
$p(\gamma.x) = \gamma.p(x) \quad \forall x \in X$ and
$d(\gamma.x, x) \geq d(\gamma.p(x), p(x)) \quad \forall x \in X$ (projections onto convex subsets are dist. non-increasing)
$\Rightarrow p(\mathrm{Min}(\gamma)) = \mathrm{Min}(\gamma) \cap C = \mathrm{Min}(\gamma|_C)$.

$\square$

III.4 Example (Hyperbolic plane)

$X = \mathbb{H}^2 = \{x+iy \mid x,y \in \mathbb{R}, y>0\} \subseteq \mathbb{C}$ upper half-plane model

hypob. metric is given by $ds^2 = \dfrac{dx^2+dy^2}{y^2}$

$SL(2,\mathbb{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} =: \gamma$ acts via

$$x+iy \xmapsto{\gamma} \frac{a(x+iy)+b}{c(x+iy)+d}$$ as an (orientation preserving) isometry.

The kernel of the action of $SL(2,\mathbb{R}) \curvearrowright X$ is $\pm \mathbb{1}$.

$\rightsquigarrow$ ~~xxxx~~

<u>Fact</u>: $\mathrm{Isom}_+(\mathbb{H}^2) = PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/_{\pm\mathbb{1}}$.
↑ orient. preserving

<u>Linear algebra classif. of isometries</u>: ↑ applies also to $PSL(2,\mathbb{R})$

$f \in SL(2,\mathbb{R})$ is (1) elliptic iff it is $\pm\mathbb{1}$ or diagonalizable in $SL(2,\mathbb{C})$ but not in $SL(2,\mathbb{R})$ (2) hyperbolic iff it is $\neq \pm\mathbb{1}$ and diagonalizable in $SL(2,\mathbb{R})$ (3) parabolic iff it is not diagonalizable in $SL(2,\mathbb{C})$.

the <u>geometric interpretation</u> is

(1) $\exists$ fixed pt in X

(2) $\nexists$ fixed pt but an invariant geodesic

(3) all others

This fits with Def III.2.

<u>III.5 Example</u> (Euclidean plane)

$AA^t = \mathbb{1}$, orthog. mx

Let $\gamma \in \mathrm{Isom}(\mathbb{R}^2, d_{eucl.})$

$\rightsquigarrow$ γ is of the form $x \mapsto Ax+b$ with $A \in O(2)$ and $b \in \mathbb{R}^2$.

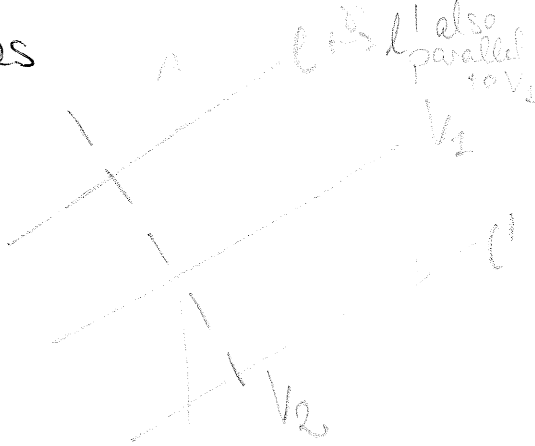
Suppose γ is not elliptic (i.e. does not fix a pt.).

Then $A-\mathbb{1}$ does not have $-b$ in its image, and is hence not invertible.

$\Rightarrow \exists v$ s.th. $Av = v$ for some $v \in \mathbb{R}^2 \setminus \{0\}$.

Put $V_1 := \mathrm{span}(v)$, $V_2 := V_1^\perp$

$\Rightarrow$ $\mathbb{R}^2 \cong V_1 \times V_2$ and γ maps all lines parallel to V_1 to such lines



$\rightsquigarrow$ One can show (using BH I.5.3(4)) that γ can be written as a product of a (non-triv.) translation in V_1 and an (elliptic) isometry of V_2.

e.g. γ a glide refl. along V_1

Similar decompositions exist for $\mathbb{R}^n$ $\rightsquigarrow$ BH II.6.5

Fact: Every isometry of $\mathbb{R}^n_{eucl}$ is semi-simple.

Exercise: Recall: $\mathbb{R}$-tree = geod. metr. space that is CAT(κ) $\forall\ \kappa \in \mathbb{R}$.

Show: every isometry of an $\mathbb{R}$-tree is semi-simple.

Hint: $x \in X \leftarrow \mathbb{R}$-tree, γ isom., $m \in [x, \gamma.x]$ midpoint. show $d(m, \gamma.m) = |\gamma|$.

We now look again at isometries of general CAT(0) spaces.

III.6 Prop. $\cancel{*}$ (characterization of elliptic isom.)

X complete CAT(0), $\gamma \in \mathrm{Isom}(X)$. Then

γ elliptic $\Leftrightarrow$ γ has a bounded orbit.

In addition, if γ^n is elliptic, then γ is elliptic.
for some $n \neq 0$

Proof: The equivalence follows from the existence of a center of bounded subset of a CAT(0) space. (see I.13)

In case γ^n is elliptic and fixes $x \in X$, then $\cancel{*}$ the γ-orbit of x is finite (and hence bounded). Therefore γ is elliptic. $\square$

III.7 Prop. (characterization of hyperbolic isom.)

X CAT(0), $\gamma \in \mathrm{Isom}(X)$, then

γ hyperbolic $\Longleftrightarrow$ $\exists$ a geodesic $c: \mathbb{R} \to X$ which is translated by γ, i.e. $\gamma . c(t) = c(t+a)$ for some $a > 0$.

Def. We call $c(\mathbb{R})$ an <u>axis</u> for γ.

One can show: $|\gamma| = a$

Proof: See BH II.6.8 (1) $\square$

<u>We collect some more facts about Min-sets:</u>

III.8 Thm (Min-sets)

X CAT(0), γ hyperbolic isometry, then

(1) $\mathrm{Min}(\gamma)$ = the union of all axes of γ. Moreover all axes are parallel.

(2) $\mathrm{Min}(\gamma) \cong Y \times \mathbb{R}$ for some space Y

and $\gamma|_{\mathrm{Min}(\gamma)}: (y,t) \longmapsto (y, t+|\gamma|)$ $y \in Y, t \in \mathbb{R}$

see BH II.6.8.

Now on to CAT(0) groups:

III.9 Def. We say that <u>a group G</u> is <u>CAT(0)</u>

if it admits an action on a CAT(0) space which is proper, cocompact and by isometries.

proper: $\forall x \in X$ $\exists$ $B_\delta(x)$ s.th.
$|\{B_\delta(x) \cap g.B_\delta(x) \neq \emptyset, g \in G\}| < \infty$

cocompact: X/G compact

Rmk ⚠ Many grps that can not act properly and cocompactly by isometries on a CAT(0) space (i.e. are not CAT(0)) embed as subgrps into grps that do.

⇝ see section 5 of Chapter III.Γ in BH for complicated subgrps of CAT(0) grps

III.10 Examples of CAT(0) groups

- Coxeter groups (of all types)
- RAAGs ⇝ given by graphs
- Small-cancellation grps
- all CAT(-1) grps, such as
  - finite grps
  - f.g. grps that are q.i. to free grps
  - $\pi_1(S_g)$, $g \geq 2$
- braid groups (for $n \leq 6$, conjectured $\forall n$)

open: are all Artin groups CAT(0)? (Charney)

We now prove a first property of CAT(0) grps:

III.11 Prop. (finite subgrps)

G a CAT(0) group, then G contains only finitely many conjugacy classes of finite subgroups.

To see this proposition, we ~~first show the following lemma~~. argue as follows.

<u>Proof</u> First observe from the BT-fixed pt thm (see I.14) that a finite grp Γ which acts on a CAT(0) space X has a non-empty, convex set of fixed points $\mathrm{Fix}(\Gamma) \subseteq X$.

In addition one can show that cocompact actions only have finitely many <u>isotropy subgrps</u>, i.e. grps $\Gamma_x = \{\gamma \in \Gamma \mid \gamma . x = x\}$. (BH I.8.5(5))

This implies the statement. $\square$

<u>Next time:</u> more properties of CAT(0) grps & boundaries