Exercise 1 (4 points)
Let $R$ be a ring, $S \subseteq R$ be a multiplicatively closed subset of $R$ such that $1 \in S$, and $M$ be an $R$-module. On $M \times S$ we define an equivalence relation by

$$(m, s) \sim (m', s') :\iff \exists t \in S : t \cdot (s m' - s' m) = 0.$$

We denote the equivalence class of $(m, s)$ by $\frac{m}{s}$. The set $(M \times S)/\sim$ is called localization of $M$ at $S$ and is denoted by $S^{-1}M$ or $M_S$. The following operations turn $M_S$ into an $R$-module ($m, m' \in M, s, s' \in S, r \in R$):

\[
\begin{align*}
\frac{m}{s} + \frac{m'}{s'} &:= \frac{s' m + s m'}{ss'} \\
r \cdot \frac{m}{s} &:= \frac{rm}{s}
\end{align*}
\]

a) Show that $M_S \cong M \otimes_R R_S$.

b) Show that, for any $R_S$-module $U$, the modules $M \otimes_R U$ and $M_S \otimes_{R_S} U$ are isomorphic.

c) Now let $N$ be another $R$-module. Show that $M_S \otimes_{R_S} N_S \cong (M \otimes_R N)_S$.

Exercise 2 (4 points)
In Exercise 1, we defined localization. Similarly to the case of localization of rings, for an $R$-module $M$ and a prime ideal $P \triangleleft R$ we denote the localization of $M$ at $R \setminus P$ by $M_P$.

Show that if an $R$-module $M$ is flat, then for all prime ideals $P \triangleleft R$, $M_P$ is flat as an $R_P$-module.

(Remark: The other direction is also true.)

Exercise 3 (4 points)
Let $R$ be a ring and

\[
\begin{array}{cccccc}
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\
\downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & & \\
0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\
\end{array}
\]

be a commutative diagram of $R$-modules with exact rows. Show that $f_2$ is injective if $f_1$ and $f_3$ are injective. Show that $f_2$ is surjective if $f_1$ and $f_3$ are surjective.
**Exercise 4** (4 points)

Let $R$ be a ring and $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ be a short exact sequence of $R$-modules where $C$ is flat. Show that:

a) For any $R$-module $M$, the induced sequence

$$0 \rightarrow A \otimes_R M \rightarrow B \otimes_R M \rightarrow C \otimes_R M \rightarrow 0$$

is exact. (Hint: Fit all exact sequences you can imagine into one big diagram and chase elements around.)

b) $A$ is flat if and only if $B$ is flat.

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**Solutions to be handed in** on Tuesday, 6.5.2008, at the beginning of the problem session in S12.