Exercise 1 (4 points)
Let $J, I_1, \ldots, I_n \subseteq R$ be ideals such that at most two of the $I_j$ are not prime. Show that if $J \subseteq I_1 \cup \ldots \cup I_n$ then $J$ is contained in one of the $I_j$.

*Hint:* Use induction on $n$ and do the case $n = 2$ separately.

Exercise 2 (4 points)
Show that $R[[X]]$ is noetherian if $R$ is noetherian.

Exercise 3 (4 points)
Let $R$ be a ring such that for any maximal ideal $m$, the localized ring $R_m$ is noetherian. Furthermore, suppose that any element $r \in R \setminus \{0\}$ is contained in only finitely many maximal ideals. Show that $R$ is noetherian.

*Hint:* To show that an ideal $I$ is finitely generated, choose an arbitrary element $x \in I \setminus \{0\}$ and consider the localizations of $R$ at all maximal ideals containing $x$.

Exercise 4 (4 points)
An $R$-module $M$ is called artinian if and only if any decreasing sequence of submodules of $M$ eventually becomes stationary. The ring $R$ is called artinian if it is artinian as $R$-module. Show that:

a) If

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

is a short exact sequence of $R$-modules, then $M$ is artinian if and only if $M'$ and $M''$ are artinian.

b) Every integral artinian ring is a field.

c) In an artinian ring, every prime ideal is maximal.

Solutions to be handed in on Tuesday, 27.5.2008, at the beginning of the problem session in S12.