Algebra II – Problem Sheet 8

Exercise 1 (4 points)
Let $S = \bigoplus_{i \in \mathbb{N}} S_i$ be a graded ring.

a) For an ideal $I \trianglelefteq S$, let $I_{\text{hom}} := \{x \in I : x \text{ homogeneous}\}$ be the subideal generated by all homogeneous elements of $I$. Show that $I_{\text{hom}}$ is prime if $I$ is prime. Is the converse also true?

b) Let $I$ be a homogeneous ideal and let $V(I)$ be the set of all prime ideals of $S$ containing $I$. Show that any minimal element of $V(I)$ is homogeneous.

Exercise 2 (6 points)
Let $R$ be a ring, $I \trianglelefteq R$ be an ideal and $M$ be an $R$-module. We define
\[ \text{gr}_I(M) := \bigoplus_{n \in \mathbb{N}} I^n M / I^{n+1} M. \]
Show that:

a) $\text{gr}_I(R)$ is a graded ring.

b) $\text{gr}_I(M)$ is a graded $\text{gr}_I(R)$-module.

c) $\text{gr}_I(R)$ is noetherian if $R$ is noetherian.

d) If $R = \mathbb{Z}$ and $I = p\mathbb{Z}$ for a prime number $p$ then $\text{gr}_I(R) \cong \mathbb{F}_p[X]$.

Exercise 3 (6 points)
Let $K$ be a field, $S = K[X,Y,Z]$ and $f \in S$ be a homogeneous polynomial of degree $d$.

a) Determine the Hilbert polynomial of $S/(f)$.

b) Determine the Hilbert series of $S$.

Solutions to be handed in on Tuesday, 10.6.2008, at the beginning of the problem session in S12.