Exercise 1 (4 points)
Let \( R \) be a local ring with maximal ideal \( m \) and \( M, N \) be finitely generated \( R \)-modules.

a) Show that \( M = 0 \iff M/mM = 0 \).

b) If \( M \neq 0 \), show that there is a surjective homomorphism \( M \otimes_R R/m \rightarrow R/m \) and hence a surjective homomorphism \( M \otimes_R N \rightarrow N/mN \).

c) Show that if \( M \otimes_R N = 0 \) then \( M = 0 \) or \( N = 0 \).

Exercise 2 (4 points)
Let \( R := \{ f = \sum_{i=0}^{d} a_i X^i \in \mathbb{Q}[X] : a_0 \in \mathbb{Z} \} \).

a) Show that \( R \) is a ring which is not noetherian.

b) Find an ideal \( I \trianglelefteq R, I \neq R \) such that \( \bigcap_{n \in \mathbb{N}} I^n \neq 0 \).

Exercise 3 (4 points)
Let \( R \) be a noetherian local ring and \( m \) be its maximal ideal.

a) Show that if there is an \( n \in \mathbb{N} \) such that \( m^n = m^{n+1} \), then \( \dim(R) = 0 \).

b) Find an example for a noetherian local ring with maximal ideal \( m \neq 0 \) in which the assumption in a) is fulfilled.
Exercise 4 (4 points)
Let \( K \) be a field and \( R := K[X_1, X_2, \ldots] \) be the polynomial ring in countably many variables. We define the following prime ideals in \( R \):
\[
P_0 := (X_1),
P_1 := (X_2),
P_2 := (X_3, X_4),
P_3 := (X_5, \ldots, X_8),
\vdots
P_n := (X_{2^n-1+1}, \ldots, X_{2^n}) \quad (n \in \mathbb{N}_{\geq 1}).
\]
Let \( U := R \setminus \bigcup_{n \in \mathbb{N}} P_n \) and \( S := R_U \).

a) Determine all maximal ideals of \( S \).

Hint: Use that for any ideal \( J \trianglelefteq S, J \neq S \), we have the following facts:

(i) \( J \cap R \subseteq \bigcup_{n \in \mathbb{N}} P_n \)
(ii) \( \exists n_0 \in \mathbb{N} : J \cap R \subseteq P_0 \cup \ldots \cup P_{n_0} \)
(iii) \( \exists k \in \mathbb{N} : J \cap R \subseteq P_k \)

b) Show that \( S \) is noetherian.

Hint: Use the fact that \( S_{P_n, S} \cong R_{P_n} \) and the fact that for any integral ring \( \overline{R} \) and any system of variables \( \mathcal{X} \) we have \( \overline{R}[\mathcal{X}]_{\langle \mathcal{X} \rangle} \cong \text{Quot}(\overline{R})[\mathcal{X}]_{\langle \mathcal{X} \rangle} \). Apply Exercise 3 on Problem Sheet 6.

c) Show that \( \dim(S) = \infty \).

Solutions to be handed in on Tuesday, 24.6.2008, at the beginning of the problem session in S12.