Exercise 1 (4 points)
Let $R$ be a local ring with maximal ideal $m$ and $M, N$ be finitely generated $R$-modules.

a) Show that $M = 0 \iff M/mM = 0$.

b) If $M \neq 0$, show that there is a surjective homomorphism $M \otimes_R R/m \to R/m$ and hence a surjective homomorphism $M \otimes_R N \to N/mN$.

c) Show that if $M \otimes_R N = 0$ then $M = 0$ or $N = 0$.

Exercise 2 (4 points)
Let $R := \{ f = \sum_{i=0}^{d} a_i X^i \in \mathbb{Q}[X] : a_0 \in \mathbb{Z} \}$.

a) Show that $R$ is a ring which is not noetherian.

b) Find an ideal $I \triangleleft R$, $I \neq R$ such that $\bigcap_{n \in \mathbb{N}} I^n \neq 0$.

Exercise 3 (4 points)
Let $R$ be a noetherian local ring and $m$ be its maximal ideal.

a) Show that if there is an $n \in \mathbb{N}$ such that $m^n = m^{n+1}$, then $\dim(R) = 0$.

b) Find an example for a noetherian local ring with maximal ideal $m \neq 0$ in which the assumption in a) is fulfilled.
Exercise 4  (4 points)

Let $K$ be a field and $R := K[X_1, X_2, \ldots]$ be the polynomial ring in countably many variables. We define the following prime ideals in $R$:

- $P_0 := (X_1)$,
- $P_1 := (X_2)$,
- $P_2 := (X_3, X_4)$,
- $P_3 := (X_5, \ldots, X_8)$,
- $\vdots$
- $P_n := (X_{2^{n-1}+1}, \ldots, X_{2^n}) \quad (n \in \mathbb{N}_{\geq 1})$.

Let $U := R \setminus \bigcup_{n \in \mathbb{N}} P_n$ and $S := RU$.

a) Determine all maximal ideals of $S$.

b) Show that $S$ is noetherian.

   *Hint:* Use the fact that $S_{P_n}S \cong R_{P_n}$. Apply Exercise 3 on Problem Sheet 6.

c) Show that $\dim(S) = \infty$.

Solutions to be handed in on Tuesday, 24.6.2008, at the beginning of the problem session in S12.