Algebra II – Problem Sheet 12

Exercise 1 (5 points)
Let \( R \) be a local ring with maximal ideal \( m \). For a finitely generated \( R \)-module \( M \), let \( \mu(M) \) be the minimal number of generators.

a) Show that \( \mu(M) = \dim_{R/m}(M/mM) \).

b) Show that in any system of generators of \( M \) there are \( \mu(M) \) elements generating \( M \).

Now let \( R \) be a local, finitely generated, integral algebra over a field \( K \) with maximal ideal \( m \). For any element \( x \in m \), let \( P_x \subseteq m \) be a minimal prime ideal containing \( x \). Use without proof that \( \text{ht}(P_x) = 1 \) if \( x \in m \setminus \{0\} \). Show that:

c) For any \( x \in m \), such a \( P_x \) always exists.

d) \( \dim(R) = \dim(R/P_x) + 1 \)

e) \( \dim(R) \leq \mu(m) \)

Exercise 2 (5 points)
Let \( R \) be a noetherian, integral domain which is not a field. Show that the following statements are equivalent:

(i) \( R \) is a discrete valuation ring.

(ii) For all \( x \in \text{Quot}(R) \) we have \( x \in R \) or \( x^{-1} \in R \).

(iii) The set of all principal ideals of \( R \) is totally ordered.

(iv) The set of all ideals of \( R \) is totally ordered.

(v) \( R \) is a local ring and a principal ideal domain.

Exercise 3 (2 points)
Show that a sequence \((a_n)_{n \in \mathbb{N}} \subseteq (\mathbb{Q}, | \cdot |_p)\) is Cauchy if and only if \( |a_{n+1} - a_n|_p \xrightarrow{n \to \infty} 0 \).
Exercise 4  (4 points)
Let $R$ be a discrete valuation ring with maximal ideal $m$. Let

$$S := \{(x, y) \in R \times R \mid x - y \in m\}.$$ 

Show that:

a) $S$ is a subring of $R \times R$.

b) $S$ is local.

c) There are exactly three prime ideals in $S$, i.e. two prime ideals which are not maximal.

Solutions to be handed in on Tuesday, 8.7.2008, at the beginning of the problem session in S12.