Contemporary Mathematics

Teichmüller curves defined by characteristic origamis

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Abstract. We study translation surfaces with Veech group $SL_2(\mathbb{Z})$. They all arise as origamis; any characteristic subgroup of F_2 provides an example. For any given origami, we construct one having Veech group $SL_2(\mathbb{Z})$ that dominates it. As an application we investigate two series of explicit examples: the Heisenberg origamis, where we obtain equations for the associated Teichmüller curves, and a sequence of origamis starting from "stairs" and related to dihedral groups.

1. Introduction

An *origami* is a closed surface X which is obtained from a finite number of euclidean squares by glueing each right edge to a left one and each top edge to a bottom one. Mapping each of the squares onto a torus E in the obvious way one obtains a covering map $p : X \to E$ which is ramified at most in the vertices of the squares. Conversely, any finite covering $p : X \to E$ of a torus E by a closed surface X, that is ramified over only one point $0 \in E$, defines an origami as above.

This way of tiling a (punctured) surface into squares and thus endowing it with a translation structure has been studied since the late seventies by many authors, including Thurston, Veech, Earle/Gardiner, Gutkin/Judge, and others. Because of the combinatorial construction above P. Lochak proposed the name "origami". It is also used in G. Schmithüsen's paper [**S1**] which we shall follow closely in notation and exposition.

Given an origami $O = (p : X \to E)$ as above, the unramified covering $p : X^* = X - p^{-1}(0) \to E^* = E - \{0\}$ corresponds to a subgroup $U(O)$ of finite index of the fundamental group of E^*. Note that $\pi_1(E^*)$ is a free group F_2 on two generators x and y, and $U(O)$ is isomorphic to $\pi_1(X^*)$. Conversely, any subgroup U of F_2 of finite index corresponds to a finite unramified covering $p : X^* \to E^*$. Any such covering can be extended in a unique way to a (ramified) covering $p : X \to E$ from a closed surface X containing X^* as a cofinite subset.

2000 *Mathematics Subject Classification.* 14H10, 14H30, 32G15.

Key words and phrases. Teichmüller curves, origamis, Veech groups, characteristic subgroups.

Definition 1.1. An origami O is called *characteristic* if $U(O)$ is a characteristic subgroup of F_2.

It is a well known fact that the characteristic subgroups of F_2 are cofinal among all subgroups of finite index, i. e. every subgroup of finite index in F_2 contains a characteristic subgroup of F_2 that still has finite index. In view of the remarks preceding Definition 1.1 we obtain

Proposition 1.2. *For every origami $O = (p : X \to E)$ there is a characteristic origami $\tilde{O} = (\tilde{p} : \tilde{X} \to E)$ dominating O, i. e. such that $\tilde{p}$ factors through p.*

In Section 2 we shall present an effective method to construct, for any given origami O, a dominating characteristic origami $\tilde{O}$. However, the construction does not necessarily yield the smallest possible dominating characteristic origami.

For any origami O, the affine diffeomorphisms of O determine a discrete subgroup $\Gamma(O)$ of $\mathrm{SL}_2(\mathbb{R})$, called the *Veech group* of O. In Section 3 we recall G. Schmithüsen's characterization of $\Gamma(O)$ in terms of automorphisms of F_2, see [**S1**]. Using this theorem it follows easily that the Veech group of any characteristic origami is equal to $\mathrm{SL}_2(\mathbb{Z})$. We also explain how variation of the translation structure leads to an algebraic curve $C(O)$ in the moduli space M_g of curves of genus g, called the *origami curve* associated with O; it is a special case of a Teichmüller curve. For a characteristic origami, $C(O)$ turns out to be isomorphic to the affine line.

In the last section of this paper we study several examples of characteristic origamis. By far the smallest nontrivial characteristic origami W has degree 8 and is related to the quaternion group. We study it in greater detail in a joint work [**HS**] with G. Schmithüsen. For any n and suitable l we find a characteristic origami $O_{n,l}$ which corresponds to the Heisenberg type group $G_{n,l}$. We exhibit an equation for the 1-parameter family of curves induced by O, and determine the cusp of the origami curve $C(O_{n,l})$. We obtain a second infinite sequence of examples by applying the construction of Section 2 to the stairlike origami St_n for odd $n \geq 3$. The characteristic origami $\widetilde{St}_3$, of degree 108, was the first (nontrivial) example explicitly constructed.

It is a pleasure for me to acknowledge the influence and contribution of Gabriela Schmithüsen to this work. Many of the ideas in this paper are due to her, or grew out of discussions with her. The discovery of the first Heisenberg origami $O_{3,3}$ resulted from a joint effort with her and Martin Möller. I would like to thank both of them heartily.

2. Construction of characteristic origamis

In this section we give a proof of Prop. 1.2, i. e. we show that every origami is dominated by a characteristic one. As explained in the previous section, this is equivalent to showing that every subgroup U of F_2 of finite index contains a characteristic subgroup $H \subseteq U \subseteq F_2$ of finite index. We shall show this more generally for any finitely generated group Γ in place of F_2.

First we recall the group theoretic argument: For any group Γ and any subgroup U of Γ of finite index, set $U_{norm} := \bigcap \gamma U \gamma^{-1}$, where γ runs through a set of coset representatives of U in Γ. U_{norm} is a normal subgroup of Γ, contained in U and still of finite index in Γ; it is the largest subgroup of Γ with this property.

To finish the proof of Prop. 1.2, let $U_{char} := \bigcap \varphi(U)$, where φ runs through $\mathrm{Aut}(\Gamma)$. Clearly U_{char} is a characteristic subgroup of Γ , contained in U_{norm} and maximal with this property. For any $\varphi \in \mathrm{Aut}(\Gamma)$, $\varphi(U)$ has the same index in Γ as U. If Γ is finitely generated, it has only finitely many subgroups of a given index; therefore U_{char} is a finite intersection of finite index subgroups and thus has finite index in Γ itself.

Now we want to give an effective version of the proof of Prop. 1.2. We shall determine U_{norm} in the special situation of this paper, where $\Gamma = F_2$, with the help of origamis. Then back in the general situation we shall construct a finite index characteristic subgroup H contained in U, but in general H will be a proper subgroup of U_{char}.

Let $U \subseteq F_2$ be a subgroup of finite index; U corresponds to an origami O. Explicitly, O can be obtained from a set $w_1, \dots, w_d$ of right coset representatives of U in F_2 as follows: take d squares with labels $Uw_1, \dots, Uw_d$ and glue them such that the right neighbour of Uw_i is Uw_ix and its top neighbour is Uw_iy (with the fixed basis x, y of F_2). This glueing can be described by two permutations σ_x and σ_y in S_d: $\sigma_x(i) = j \Leftrightarrow Uw_ix = Uw_j$, and similarly for σ_y. Now let $h_U : F_2 \to S_d$ be the homomorphism that maps x to σ_x and y to σ_y. Then by construction, $\ker(h_U) \subseteq U$, and $U_{norm} = \ker(h_U)$. The factor group F_2/U_{norm} is isomorphic to the subgroup G_U of S_d generated by σ_x and σ_y. This means that we can obtain the origami O_{norm} corresponding to U_{norm} by the construction above applied to the elements of G_U as coset representatives. In particular, the origami map $p_{norm} : O_{norm} \to E$ is a normal covering with Galois group G_U.

For the construction of a characteristic subgroup H of finite index contained in U we apply

Proposition 2.1. *Let Γ be a finitely generated group, G a finite group and $X := \mathrm{Hom}_{surj}(\Gamma, G)$ the set of surjective group homomorphisms $\Gamma \to G$.*
Then we have

a) *X is finite.*

b) *Assume X is nonempty and let $\mathrm{Aut}(G)$ act on X from the right.*
Let $h_1, \dots, h_k$ be a set of representatives of the orbit set $X/\mathrm{Aut}(G)$ and let $h : \Gamma \to G^k, \gamma \mapsto (h_1(\gamma), \dots, h_k(\gamma))$ be the diagonal homomorphism.
Then $H := \ker(h)$ is a characteristic subgroup of Γ of finite index.

Proof. a) is obvious, since Γ is finitely generated and G is finite.

For b) let $\varphi \in \mathrm{Aut}(\Gamma)$ and $\gamma \in H$, i. e. $h_i(\gamma) = 1$ for all i. For $i = 1, \dots, k$, $h_i \circ \varphi : \Gamma \to G$ is again a surjective homomorphism. Therefore there is an index $j(i) \in \{1, \dots, k\}$ and $\sigma_i \in \mathrm{Aut}(G)$ such that

$$h_i \circ \varphi = \sigma_i \circ h_{j(i)}.$$

It follows that $h_i(\varphi(\gamma)) = \sigma_i(h_{j(i)}(\gamma)) = 1$ for $i = 1, \dots, k$, thus $\varphi(\gamma) \in H$. So we have $\varphi(H) \subseteq H$ for all $\varphi \in \mathrm{Aut}(\Gamma)$. Applying φ^{-1} we see that $H \subseteq \varphi^{-1}(H)$ for all $\varphi \in \mathrm{Aut}(\Gamma)$. Together this gives $\varphi(H) = H$. □

Remarks. **1)** In the situation of this paper, i. e. $\Gamma = F_2$, the set X is nonempty if and only if G can be generated by 2 elements.

2) The index of H in Γ is the order of the subgroup $\mathrm{im}(h) \subseteq G^k$. It can be small compared to the order of G^k, as the following example illustrates:
Let $G = \mathbb{Z}/n\mathbb{Z}$ ($n \geq 2$). A homomorphism $g : F_2 \to \mathbb{Z}/n\mathbb{Z}$ is surjective if and only if $g(x)$ and $g(y)$ are coprime modulo n (for the fixed generators x and y of F_2). Let $h_1, \dots, h_k$ be a set of representatives of such homomorphisms modulo $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ and $h : F_2 \to (\mathbb{Z}/n\mathbb{Z})^k$ the diagonal homomorphism as in the Proposition. Then in general k is quite large (e. g. the homomorphisms $h_i, i = 0, \dots, n-1$ given by $h_i(x) = \bar{1}$, $h_i(y) = \bar{i}$, are all inequivalent mod $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$). But the image of h in $(\mathbb{Z}/n\mathbb{Z})^k$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^2$: namely, $\mathrm{im}(h)$ is generated by $h(x)$ and $h(y)$, both of which are elements of order n; since $\mathrm{im}(h)$ is abelian, it follows that it is a factor group of $(\mathbb{Z}/n\mathbb{Z})^2$, and it is easy to see that it is not a proper factor. For later purpose we give a name to the corresponding origami:

Example 2.2. Let Tr_n be the origami that can be described as a large square which is composed of n^2 unit squares; the two horizontal and the two vertical edges of the large square are glued. The resulting surface is the torus E itself, and the origami map $p : E \to E$ is multiplication by n (if the torus is endowed with the usual group structure $\mathbb{R}^2/\mathbb{Z}^2$).

3. Veech groups of origamis

Let $O = (p : X \to E)$ be an origami. We can use the squares out of which X is built as chart maps and thus obtain a translation structure on X^* (and also on E^*). With respect to this structure, p is a translation covering.

Let $\mathrm{Aff}^+(O)$ be the group of orientation preserving diffeomorphisms of X^* that are affine with respect to the translation structure. In local coordinates such a diffeomorphism f is given by $z \mapsto Az+b$ with a matrix $A \in \mathrm{SL}_2(\mathbb{R})$ and a translation vector $b \in \mathbb{R}^2$. Since the transition maps between different local coordinates are translations, the matrix $A = A_f$ is the same on all charts and depends only on f. In this way we obtain a group homomorphism

$$\mathrm{der} : \mathrm{Aff}^+(O) \to \mathrm{SL}_2(\mathbb{R}).$$

The image $\Gamma(O) := \mathrm{der}(\mathrm{Aff}^+(O)) \subset \mathrm{SL}_2(\mathbb{R})$ is called the *Veech group* of O. Veech showed that $\Gamma(O)$ is a discrete subgroup in the more general situation of a translation structure on a (Riemann) surface that is induced by a holomorphic quadratic differential q, see [**V**, Prop. 2.7]. In the case of an origami, $q = (p^*\omega_E)^2$ with the invariant holomorphic differential ω_E on E.

G. Schmithüsen has given the following description of $\Gamma(O)$, see [**S1**, Sect. 2]: First note that any affine diffeomorphism f of X^* lifts via the universal covering $u : \mathbb{H} \to X^*$ to a diffeomorphism $\hat{f}$ of $\mathbb{H}$ which is affine with respect to the translation structure on $\mathbb{H}$ induced from X^* via u (or from E^* via $v := p \circ u$). Then $\hat{f}$ acts on $F_2 = \mathrm{Gal}(\mathbb{H}/E^*) \subset \mathrm{Aut}(\mathbb{H})$ by conjugation. This gives a group homomorphism $\alpha : \{\hat{f} \in \mathrm{Aff}^+(\mathbb{H}) : \hat{f} \text{ descends to } X^*\} \to \mathrm{Aut}^+(F_2)$. Schmithüsen shows that the image of α is $\{\varphi \in \mathrm{Aut}^+(F_2) : \varphi(U) = U\} =: \mathrm{Stab}(U)$ for the subgroup $U = U(O) = \mathrm{Gal}(\mathbb{H}/X^*)$ of $F_2 = \mathrm{Gal}(\mathbb{H}/E^*)$ and that the composition of α with the natural map $\beta : \mathrm{Aut}^+(F_2) \to \mathrm{Out}^+(F_2) \cong \mathrm{SL}_2(\mathbb{Z})$ is the homomorphism der introduced above. This gives

THEOREM 3.1 ([**S1**], Prop. 1). *For an origami O, the Veech group is the image*

$$\Gamma(O) = \beta(\mathrm{Stab}(U(O)) \subset \mathrm{SL}_2(\mathbb{Z}).$$

An immediate consequence of this theorem is the fact, first proved by Gutkin and Judge [**GJ**], that the Veech group of an origami is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index: This follows because $U(O)$ has finite index in F_2 and therefore $\mathrm{Stab}(U(O))$ has finite index in $\mathrm{Aut}^+(F_2)$. Another direct consequence is

COROLLARY 3.2. *The Veech group of a characteristic origami is* $\mathrm{SL}_2(\mathbb{Z})$.

By definition, for a characteristic origami $\mathrm{Stab}(U(O)) = \mathrm{Aut}^+(F_2)$. Conversely, if $\Gamma(O) = \mathrm{SL}_2(\mathbb{Z})$ and O is normal, then it is a characteristic origami; this holds because then $\mathrm{Stab}(U(O))$ contains the kernel of β and preimages of generators of the image of β, thus is equal to $\mathrm{Aut}^+(F_2)$. However, if O is not normal, this is no longer true: M. Schmoll has found an origami which is not characteristic, but has Veech group $\mathrm{SL}_2(\mathbb{Z})$.

For the definition of the Veech group we endowed the origami O with the translation structure obtained by identifying each square with the unit square in $\mathbb{R}^2$. By composing these chart maps with a matrix in $\mathrm{SL}_2(\mathbb{R})$ we obtain a new translation structure, and in general also a different structure as Riemann surface. More precisely, this construction gives us an embedding ρ_O of $\mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R}) = \mathbb{H}$ into the Teichmüller space $T_{g,n}$, where g is the genus of X and $n = |p^{-1}(0)|$. This construction is explained in more detail e. g. in the paper [**EG**] of Earle and Gardiner, in particular in Sect. 5. It turns out that ρ_O is an isometry with respect to the hyperbolic metric on $\mathbb{H}$ and the Teichmüller metric on $T_{g,n}$. Such a geodesic embedding is called a *Teichmüller disk*. Moreover ρ_O is equivariant for the actions of the Veech group $\Gamma(O)$ on $\mathbb{H}$ and the Teichmüller modular group $\mathrm{Mod}_{g,n}$ on $T_{g,n}$. Let $\pi : T_{g,n} \to M_g$ (the moduli space of curves of genus g) be the composition of the quotient map $T_{g,n} \to M_{g,n} = T_{g,n}/\mathrm{Mod}_{g,n}$ and the forgetful map $M_{g,n} \to M_g$, and let $C(O) := \pi(\rho_O(\mathbb{H}))$. Then $\pi : \rho_O(\mathbb{H}) \to C(O)$ factors through $\mathbb{H}/\Gamma(O)$, which by Sect. 2 is a finite covering of $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) = \mathbb{A}^1_{\mathbb{C}}$, thus in particular an affine algebraic curve. The induced map $\mathbb{H}/\Gamma(O) \to C(O)$ can be shown to be birational. Thus $C(O)$ is also an affine algebraic curve, called the *origami curve* associated with O, and $\mathbb{H}/\Gamma(O)$ is its normalization. From Cor. 3.2 we conclude

COROLLARY 3.3. *Let O be a characteristic origami of genus g. Then the normalization of $C(O)$ is isomorphic to $\mathbb{A}^1_{\mathbb{C}}$.*
In particular, the closure $\overline{C(O)}$ has exactly one cusp (i. e. point in $\overline{M}_g - M_g$).

We finish this section with a useful observation:

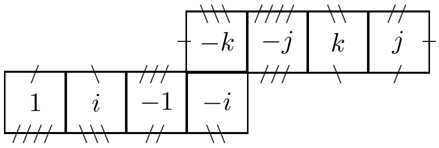
REMARK 3.4. *If $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma(O)$ for an origami $O = (p : X \to E)$ then X has an affine automorphism σ that descends via p to the canonical involution on E. σ is bianalytic for all translation structures on X^* as above. In other words, $C(O)$ is contained in the locus M_g^σ of curves in M_g "with an automorphism σ".*

PROOF. By definition, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma(O)$ implies that there is $\sigma \in \mathrm{Aff}^+(O)$ with $\mathrm{der}(\sigma) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. In local coordinates on the squares, σ acts as $z \mapsto -z$ up to translation. This obviously is holomorphic. □

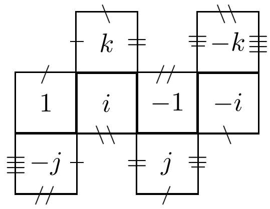
4. Examples

4.1. The quaternion origami. This is a truly remarkable origami whose extraordinary properties are studied in a joint paper [**HS**] with Gabriela Schmithüsen. Here we only give the description and collect some properties; for details and further results we refer to [**HS**].

Let Q be the quaternion group of order 8; as usual we denote its elements by $\pm1, \pm i, \pm j$ and $\pm k$. Recall that $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. Let W be the origami corresponding to Q with respect to the generators i and j, or equivalently to the kernel H of the homomorphism $h : F_2 \to Q$ given by $h(x) = i$ and $h(y) = j$. Applying the construction of Sect. 2 we see that W looks as follows:



or

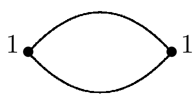


Proposition 4.1. *W is a characteristic origami.*

Proof. In view of Prop. 2.1 it suffices to show that $\mathrm{Aut}(Q)$ acts transitively on the set $\{(a, b) \in Q^2 : a, b \text{ generate } Q\} = \{(a, b) : a \in Q - \{\pm1\},\ b \in Q - \{\pm1, \pm a\}\}$. This is an elementary exercise. □

Here is a short summary of properties of W:

- The genus of W is 3.
- The origami map $p : W \to E$ factors as $p = [2] \circ p_1$, where $[2]$ is multiplication by 2 on the elliptic curve E (with suitably chosen origin) and the degree 2 covering $p_1 : W \to E$ is the quotient by the subgroup $\{\pm1\}$ of Q.
- The full automorphism group G of W is a degree two extension of Q, cf. Rem. 3.4; the center Z of G is a cyclic group of order 4.
- W/Z has genus 0. From this cyclic covering $W \to \mathbb{P}^1$ we determine the 1-parameter family of genus 3 curves that gives the origami curve $C(W)$ in M_3: it is the family
$$y^4 = x(x-1)(x-\lambda), \quad \lambda \in \mathbb{P}^1 - \{0, 1, \infty\}.$$
- The unique singular stable curve $W_\infty \in \overline{C(W)} \subset \overline{M}_3$ has two irreducible components, which intersect in two points; both components are isomorphic to the elliptic curve $y^2 = x^3 - x$.



4.2. Heisenberg origamis. Here we study a series $O_{n,l}$ of characteristic origamis that arise as special coverings of the "trivial" origamis Tr_n introduced in Example 2.2. Therefore the origami map $p : O_{n,l} \to E$ decomposes as $p = [n] \circ p_1$, where $[n]$ is multiplication by n on E and p_1 is a covering of degree l that is totally ramified over all n-torsion points of E.

We first describe the origamis $O_{n,l}$ in terms of their Galois groups $G_{n,l}$:

DEFINITION 4.2. For $n \geq 2$ and l a divisor of n let $G_{n,l}$ be the group with presentation

$$G_{n,l} = < a, b : a^n = b^n = c^l = 1, c = aba^{-1}b^{-1}, ac = ca, bc = cb >$$

We call $G_{n,l}$ a group of *Heisenberg type*.

For $l = n$, $G_n := G_{n,n}$ is isomorphic to the Heisenberg group of unipotent upper triangular matrices with entries in $\mathbb{Z}/n\mathbb{Z}$:

$$G_n = \{ \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{Z}/n\mathbb{Z}\} \subset \mathrm{SL}_3(\mathbb{Z}/n\mathbb{Z})$$

A general group of Heisenberg type is thus a quotient of some G_n by a central subgroup. Note that the relations $a^n = b^n = 1$, together with the requirement that the commutator $c = aba^{-1}b^{-1}$ be in the center, already imply $c^n = 1$ (by induction on k we find $c^k = a^k ba^{-k}b^{-1}$). Therefore we only get groups $G_{n,l}$ for $l|n$.

Let $H_{n,l}$ be the kernel of the homomorphism $h_{n,l} : F_2 \to G_{n,l}$, $x \mapsto a$, $y \mapsto b$, and $O_{n,l}$ the origami corresponding to $H_{n,l}$.

PROPOSITION 4.3. **a)** *For any $n \geq 2$ and $l|n$ there is a short exact sequence*

$$1 \to \mathbb{Z}/l\mathbb{Z} \to G_{n,l} \to (\mathbb{Z}/n\mathbb{Z})^2 \to 1.$$

In particular, the order of $G_{n,l}$ is $n^2 \cdot l$.

b) *$O_{n,l}$ is characteristic if and only if n is odd or n is even and $l|\frac{n}{2}$.*

PROOF. a) The subgroup $<c> \cong \mathbb{Z}/l\mathbb{Z}$ of $G_{n,l}$ is central, hence normal. The factor group is $< \bar{a}, \bar{b} : \bar{a}^n = \bar{b}^n = 1, \bar{a}\bar{b} = \bar{b}\bar{a} > \cong (\mathbb{Z}/n\mathbb{Z})^2$.

b) First observe that

$$ba = abc^{-1}, \tag{4.1}$$

because $b^{-1}a^{-1}ba = b^{-1}a^{-1}c^{-1}ab = c^{-1}$ since c is central.

Therefore any element of $G_{n,l}$ can uniquely be written in the form $g = a^i b^j c^\lambda$ with $0 \leq i, j < n$, $0 \leq \lambda < l$.

From (4.1) we find by induction that for any $k \geq 0$

$$g^k = (a^i b^j c^\lambda)^k = a^{ik} b^{jk} c^{\lambda k - ij\binom{k}{2}}. \tag{4.2}$$

In particular, $g^n = c^{-ij\binom{n}{2}}$. If $l|\binom{n}{2}$, this implies $g^n = 1$ for every $g \in G_{n,l}$. Note that, for odd n, this condition is automatically satisfied since l divides n; for even n, it is satisfied if and only if $l|\frac{n}{2}$.

Now assume $l|\binom{n}{2}$ and let s, t be generators of $G_{n,l}$. Their images in $(\mathbb{Z}/n\mathbb{Z})^2$ generate this group, therefore s and t both have order n. Since $G_{n,l}/ < c >$ is abelian, c generates the commutator subgroup of $G_{n,l}$. Thus $\tilde{c} := sts^{-1}t^{-1} = c^m$ for some m; this shows that $\tilde{c}^l = 1$ and that s and t both commute with $\tilde{c}$.

We have shown that the map $a \mapsto s$, $b \mapsto t$ respects all defining relations of $G_{n,l}$ and thus induces an endomorphism σ of $G_{n,l}$. Since σ is surjective, it is an automorphism. Therefore all surjective homomorphisms $F_2 \to G_{n,l}$ are equivalent in the sense of Prop. 2.1, and thus $H_{n,l}$ is characteristic.

Conversely, if $l \not| \binom{n}{2}$ we have $(ab)^n = c^{-\binom{n}{2}} \neq 1$ (more precisely ab has order $2n$ in this case). Therfore the automorphism φ_T of F_2, that sends x to xy and y to y, maps $x^n \in H_{n,l}$ to $(xy)^n \notin H_{n,l}$, showing that $H_{n,l}$ is not characteristic. □

The calculations in the above proof also allow us to describe the origami $O_{n,l}$ corresponding to $G_{n,l}$: It consists of l large squares of $n \times n$ unit squares each; we call the large squares the *leaves* of $O_{n,l}$. The individual unit squares are labeled (i, j, λ), $0 \leq i, j < n$, $0 \leq \lambda < l$ (corresponding to the elements $a^i b^j c^\lambda$ of $G_{n,l}$). Here i denotes the column, j the row, and λ the leaf of the square. The glueing comes from right multiplication by a and b: In vertical direction, (i, j, λ) is glued to $(i, j+1, \lambda)$ (where $j+1$ has to be taken mod n), i. e. to the vertical neighbour on the same leaf. In horizontal direction, (i, j, λ) is glued to $(i+1, j, \lambda - j)$. In other words: when going one square to the right, the leaf is changed, and the amount of change is given by the row number.

We list some properties of $O_{n,l}$:

Proposition 4.4. *Let $n \geq 2$ and l a divisor of n and of $\binom{n}{2}$.*
a) *$O_{n,l}$ has genus $g_{n,l} = \frac{1}{2}\, n^2(l-1) + 1$.*

b) *The unique singular stable curve $C^\infty_{n,l}$ in $\overline{C(O_{n,l})}$ has n irreducible components $C_1, \ldots, C_n$, each of genus $\frac{1}{2}(l-1)(n-2)$. C_i and C_j intersect in l points if $i - j \equiv \pm 1 \bmod n$, and are disjoint otherwise.*

c) *The automorphism group $\widetilde{G}_{n,l}$ of $O_{n,l}$ contains an element $\tilde{c}$ of order $2l$ with $\tilde{c}^2 = c$. The quotient curve $O_{n,l}/ < \tilde{c} >$ has genus 0. $C(O_{n,l})$ is induced by the 1-parameter family of curves*

$$y^{2l} = x(x-1)^l(x+1)^l(x-\lambda)^l \cdot \prod_{i=1}^{(n^2-1)/2} (x-\alpha_i)^2 \quad (n \text{ odd})$$

$$y^{2l} = x(x-1)(x+1)(x-\lambda) \cdot \prod_{i=1}^{(n^2-4)/2} (x-\alpha_i)^2 \quad (n \text{ even}).$$

Here $\lambda \in \mathbb{P}^1 - \{0, 1, \infty\}$, and the α_i are the x-coordinates of the n-torsion points on the elliptic curve $y^2 = x(x-1)(x-\lambda)$, that are not 2-torsion points.

Proof. a) The covering $p_1 : O_{n,l} \to E = O_{n,l}/{<c>}$ is totally ramified over all vertices of the small squares, i. e. over all n-torsion points. By Riemann-Hurwitz, $2g_{n,l} - 2 = n^2(l-1)$.

b) The origami curve $\overline{C(O_{n,l})}$ has only one cusp by Cor. 3.3. The corresponding stable curve $C^\infty_{n,l}$ can be obtained, e. g., by contracting the multicurve which consists of the centers of the vertical cylinders. These lines divide each of the small squares into two rectangles of height 1 and length $\frac{1}{2}$. The irreducible component C_i consists of the right halves of the squares (i, j, λ) and the left halves of $(i+1, j, \lambda)$ for all j and λ. On each leaf there is a line separating C_i from C_{i+1}, so these components intersect in l points, and each of the contracted lines is of this form.

Since $G_{n,l}$ acts transitively on the irreducible components, they all have the same genus $\bar{g}$, and thus the arithmetic genus of $C^\infty_{n,l}$ is $n \cdot \bar{g} + n \cdot l - (n-1)$. This has to be equal to $g_{n,l}$, from which we deduce $\bar{g} = \frac{1}{2}(l-1)(n-2)$.

c) By Rem. 3.4, $O_{n,l}$ has an automorphism σ with $\mathrm{der}(\sigma) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. On the squares it can (up to translation) be realized as rotation by π about the center; we may assume that σ fixes the center of the square $(0,0,0)$. Then $\tilde{c} := b^{-1}a^{-1}\sigma$ fixes the vertex $(0,0)$ at the bottom left corner of that square (and thus of leaf 0). $\tilde{c}$ maps the square (i,j,λ) to $(n-1-i, n-1-j, \lambda')$ for suitable λ'. In particular, $\tilde{c}^2$ fixes all vertices of the small squares, and it interchanges the leaves in the same order as c.
If l is even, $\tilde{c}$ fixes also the vertices $(\frac{n}{2}, 0)$, $(0, \frac{n}{2})$ and $(\frac{n}{2}, \frac{n}{2})$. Thus $\tilde{c}$ has 4 fixed points of order $2l$ and $n^2 - 4$ fixed points of order l. By Riemann-Hurwitz we find for the genus $\tilde{g}$ of $O_{n,l}/<\tilde{c}>$

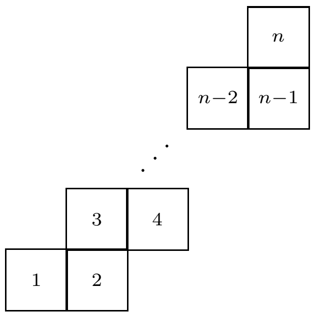
$$2g_{n,l} - 2 = n^2(l-1) = -2l(2\tilde{g} - 2) + 4 \cdot (2l-1) + (n^2 - 4)(l-1),$$

which gives $\tilde{g} = 0$.
If l is odd, $(0,0)$ is the only fixed point of $\tilde{c}$. But in this case, $\tilde{c}^l$ fixes the points $(\frac{n}{2}, 0, \lambda)$, $(0, \frac{n}{2}, \lambda)$ and $(\frac{n}{2}, \frac{n}{2}, \lambda)$ (for all λ), which are the midpoints of the bottom edge of the square $(\frac{n-1}{2}, 0, \lambda)$, the left edge of $(0, \frac{n-1}{2}, \lambda)$, and the center of the square $(\frac{n-1}{2}, \frac{n-1}{2}, \lambda)$, resp. Thus in this case , the Riemann-Hurwitz formula yields

$$n^2(l-1) = -2l(2\tilde{g} - 2) + 2l - 1 + (n^2 - 1)(l-1) + 3l,$$

and again $\tilde{g} = 0$.
The quotient map $q : O_{n,l} \to O_{n,l}/<\tilde{c}> = \mathbb{P}^1$ decomposes as $q = q_1 \circ q_2$, with $q_2 : O_{n,l} \to O_{n,l}/<c> = E$ and $q_1 : E \to \mathbb{P}^1$ the quotient by the involution $[-1]$; if E is represented in Legendre form $y^2 = x(x-1)(x-\lambda)$, q_1 is the map $(x,y) \mapsto x$. Therfore the critical values of q are the x-coordinates of the corresponding points on E (which are all n-torsion points). This shows that $O_{n,l}$ has an equation as claimed. □

4.3. Odd stairs. Let $n \geq 3$ be odd and St_n the origami



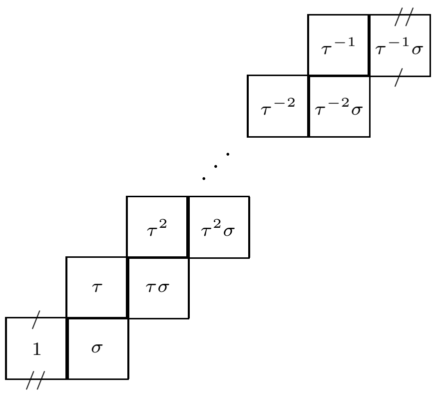
where opposite edges are glued in horizontal and in vertical direction. This is a sequence of origamis which are not normal. The smallest origami in this sequence, St_3, is also the smallest L-shaped origami $L_{2,2}$; for more information about L-shaped origamis, see [**S1**, Sect. 4] or [**HL**]. With the help of the picture it is easy to verify that all vertices of the squares are glued to a single point on St_n. This implies that the genus g_n of St_n satisfies $2g_n - 2 = n - 1$, or $g_n = \frac{n+1}{2}$. In [**S2**],

G. Schmithüsen determines the Veech group of St_n and shows in particular that it is the same for all n.

Here we shall apply the algorithm described in Sect. 2 to find a sequence $\widetilde{St}_n$ of characteristic origamis dominating St_n. In the first step we determine the smallest normal origami NSt_n dominating St_n. For this we describe St_n by the horizontal and the vertical permutation σ_a and σ_b of the squares:

$$\sigma_a = (1\,2)\,(3\,4)\,\ldots,\,(n{-}2\;n{-}1), \quad \sigma_b = (2\,3)\,(4\,5)\,\ldots,\,(n{-}1\;n).$$

Numbering the vertices of a regular n-gon suitably it is easily seen that σ_a and σ_b generate the dihedral group D_n of order $2n$. Thus NSt_n is the origami of D_n with respect to the generators σ_a and σ_b. It has $2n$ squares and again looks stairlike:



Note that the last stair is glued to the first one as indicated; we denote the elements of D_n by $\tau^i\sigma^\varepsilon$ with $0 \le i \le n-1$, $\varepsilon = 0$ or 1, where $\sigma := \sigma_a$ and $\tau := \sigma_a\sigma_b$. Again we can read off from the picture that the vertices of the squares map to exactly two different points on NSt_n, showing that its genus is $g_n^{(n)} = n$. Alternatively, we can use the fact that the commutator $\sigma_a\sigma_b\sigma_a^{-1}\sigma_b^{-1} = \tau^2$ of the generators has order n to see that the vertices fall into $4 \cdot 2n : 4 \cdot n = 2$ orbits.

PROPOSITION 4.5. *For any $n \ge 3$, there is a characteristic origami $\widetilde{St}_n$ of degree $4n^3$ and genus $2n^2(n-1)+1$ dominating St_n. The Galois group of $\widetilde{St}_n$ is*

$$K_n := \{(\delta_1, \delta_2, \delta_3) \in D_n^3 : e(\delta_1) + e(\delta_2) + e(\delta_3) = 0\},$$

where $e : D_n \to \mathbb{Z}/2\mathbb{Z}$ is the homomorphism given by $e(\tau^i\sigma^\varepsilon) = \varepsilon$.

PROOF. Let $g : F_2 \to D_n$ be a surjective homomorphism. There are the following possibilities:
1. $g(x) = \tau^i\sigma$ for some $0 \le i \le n-1$, $g(y) = \tau^j$ for some $1 \le j \le n-1$
2. $g(x) = \tau^j$ for some $1 \le j \le n-1$, $g(y) = \tau^i\sigma$ for some $0 \le i \le n-1$
3. $g(x) = \tau^i\sigma$ for some $0 \le i \le n-1$, $g(y) = \tau^j\sigma$ for some $j \ne i$.

Up to an automorphism of D_n, g is equivalent to $h_1 : x \mapsto \sigma$, $y \mapsto \tau$ in the first case, $h_2 : x \mapsto \tau$, $y \mapsto \sigma$ in the second case, and $h_3 : x \mapsto \sigma$, $y \mapsto \tau\sigma$ in the third case. The diagonal homomorphism $h = (h_1, h_2, h_3) : F_2 \to D_n^3$ is thus given by

$$h(x) = (\sigma, \tau, \sigma) =: a, \quad h(y) = (\tau, \sigma, \tau\sigma) =: b.$$

By Prop. 2.1, $HSt_n := \ker(h)$ is a characteristic subgroup of F_2 and the corresponding origami $\widetilde{St}_n$ dominates St_n.

The Galois group of $\widetilde{St}_n$ is the image of h, i. e. the subgroup of D_n^3 generated by a and b. Clearly, a and b are in K_n. Conversely, observe that $a^2 = (1, \tau^2, 1)$, $b^2 = (\tau^2, 1, 1)$ and $(ab)^2 = (1, 1, \tau^2)$. Thus $\mathrm{im}(h)$ contains any element of the form $(\tau^{i_1}, \tau^{i_2}, \tau^{i_3})$, i. e. the kernel of the homomorphism $e_3 := (e, e, e) : D_n^3 \to (\mathbb{Z}/2\mathbb{Z})^3$. Furthermore $e_3(a) = (1, 0, 1)$, $e_3(b) = (0, 1, 1)$ and $e_3(ab) = (1, 1, 0)$, which shows

$$\mathrm{im}(h) = e_3^{-1}(\{(0, 0, 0), (1, 0, 1), (0, 1, 1), (1, 1, 0)\}) = K_n.$$

In particular, the degree of $\widetilde{St}_n$ is $|K_n| = \frac{1}{2}|D_n|^3 = 4n^3$.

Finally, the commutator $aba^{-1}b^{-1} = (\tau^{-2}, \tau^2, \tau^2)$ is of order n. Therefore the vertices of the squares fall into $4n^3 : n = 4n^2$ orbits, and the genus of $\widetilde{St}_n$ is $\frac{1}{2}(4n^3 - 4n^2) + 1 = 2n^2(n - 1) + 1$. □

From the explicit description of K_n it is (in principle) possible to draw the origami $\widetilde{St}_n$ for all n. We confine ourselves here to the picture of $\widetilde{St}_3$ on the next page (see Figure 1). Some further interesting properties of $\widetilde{St}_3$ and its associated Teichmüller curve can be found in O. Bauer's Diplomarbeit [**B**].

Finally, we can also determine the stable curve $\widetilde{C}_n^\infty$ corresponding to the unique cusp of $\widetilde{St}_n$:

Proposition 4.6. *The intersection graph of $\widetilde{C}_n^\infty$ is the complete (n, n)-bipartite graph with every edge doubled. Each of the $2n$ irreducible components of $\widetilde{C}_n^\infty$ is a nonsingular curve of genus $(n - 1)^2$.*

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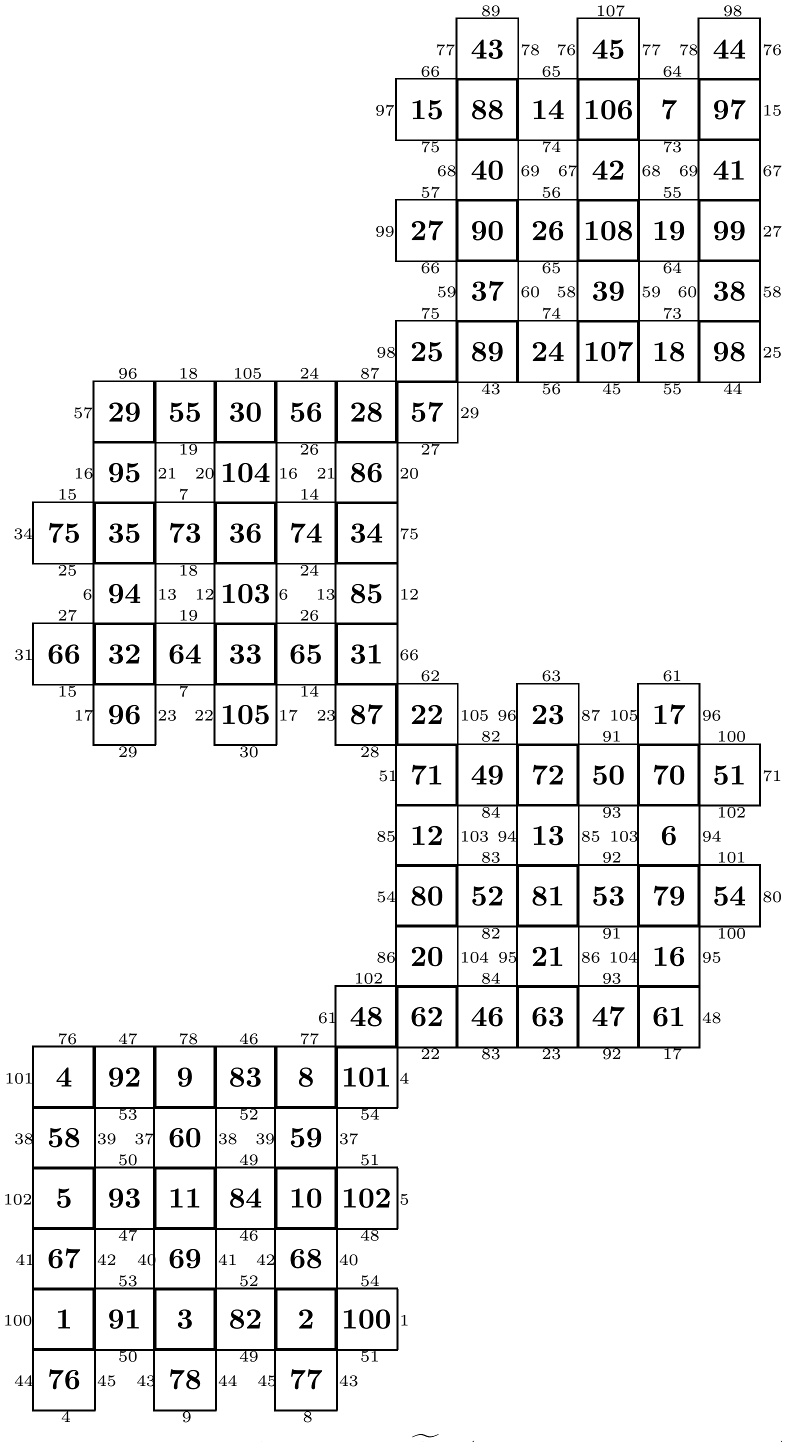

Figure 1. The 108 squares of the origami $\widetilde{St}_3$ (labeled with large numbers). The small numbers indicate the glueing of the edges.

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