Uniform dessins on Shimura curves

Jürgen Wolfart

joint work with Ernesto Girondo Sirvent and David Torres–Teigéll, UAM Madrid

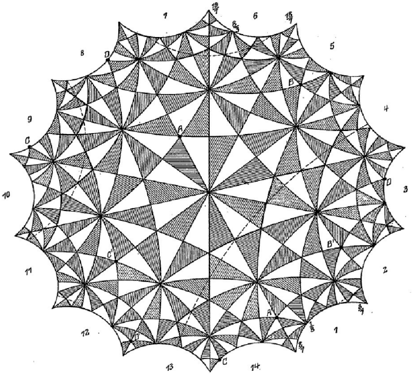
Math. Zeitschrift 2011 + work in progress

Mathematisches Seminar, Goethe–Universität Frankfurt

KIT, December 2011

Outline

Klein's quartic (I)



Fundamental domain for the covering group (= surface group) Γ of Klein's quartic $Q\ :\ x^3y+y^3z+z^3x=0$ in the hyperbolic plane $\mathbf{H}$.

Triangle groups and Belyĭ functions

Also visible in the picture: Γ is subgroup of a triangle group, here of the group $\Delta(2,3,7)$. The canonical projection

$$Q = \Gamma\backslash\mathbf{H} \to \Delta(2,3,7)\backslash\mathbf{H} \cong \mathbf{P}^1(\mathbf{C})$$

defines a Belyĭ function β: meromorphic, non–constant and ramified above three points only.

Fact 1 : all Belyĭ functions on compact Riemann surfaces come from triangle groups in this way.

Triangle groups and Belyĭ functions

Also visible in the picture: Γ is subgroup of a triangle group, here of the group $\Delta(2,3,7)$. The canonical projection

$$Q = \Gamma\backslash\mathbf{H} \to \Delta(2,3,7)\backslash\mathbf{H} \cong \mathbf{P}^1(\mathbf{C})$$

defines a Belyĭ function β: meromorphic, non–constant and ramified above three points only.

Fact 1 : all Belyĭ functions on compact Riemann surfaces come from triangle groups in this way.

Dessins d'enfants

Fact 2 : on a compact Riemann surface X there is a Belyĭ function if and only if — as an algebraic curve — X can be defined over a number field (Belyĭ 1979).

We may assume that $0, 1, \infty$ are the critical values of the Belyĭ function β . Then the β–preimage of the real interval $\circ\!\!-\!\!\!-\!\!\!-\!\!\bullet$ between 0 and 1 forms a bipartite graph cutting the Riemann surface in simply connected cells, a dessin d'enfant (Grothendieck 1984).

Here is an illustration, indicating as well the link to triangle groups:

Dessins d'enfants

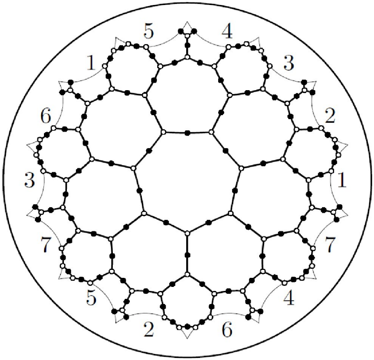
Fact 2 : on a compact Riemann surface X there is a Belyǐ function if and only if — as an algebraic curve — X can be defined over a number field (Belyǐ 1979).

We may assume that $0, 1, \infty$ are the critical values of the Belyǐ function β. Then the β–preimage of the real interval $\circ\!\!-\!\!\!-\!\!\!-\!\!\bullet$ between 0 and 1 forms a bipartite graph cutting the Riemann surface in simply connected cells, a dessin d'enfant (Grothendieck 1984).

Here is an illustration, indicating as well the link to triangle groups:

Klein's quartic (II)

Another look on the Belyĭ function for Klein's quartic Q, now: its dessin.
The numbers indicate the necessary identifications on the border.



Dessins and conformal structures

As Grothendieck pointed out, dessins induce moreover the Riemann surface structure.

Fact 3 : on the other hand, every dessin on a compact oriented 2–manifold X defines a unique conformal structure on X such that the dessin belongs to some Belyĭ function on X (Grothendieck, Singerman 1974).

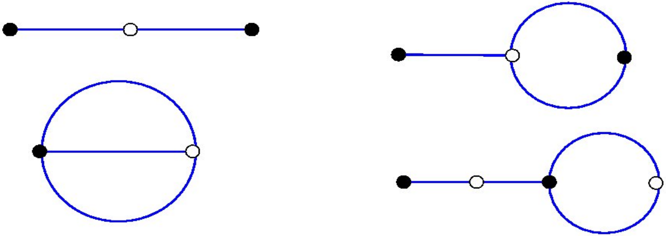
Dessins and conformal structures

As Grothendieck pointed out, dessins induce moreover the Riemann surface structure.

Fact 3 : on the other hand, every dessin on a compact oriented 2–manifold X defines a unique conformal structure on X such that the dessin belongs to some Belyĭ function on X (Grothendieck, Singerman 1974).

A converse?

No. If a dessin exists on X , it is not at all unique. Simplest example: $\mathbf{P}^1$



At least, can we explain how the different dessins on the same curve are linked to each other?
And is it possible that on the same surface coexist several non–isomorphic dessins of the same type?

Regular dessins

Restriction to more "comfortable" classes of dessins: a dessin D is called regular if there is an automorphism group acting transitively on the edges and preserving incidence, orientation and colours. Automatically, it acts as an automorphism group of the algebraic curve X as well. (For simplicity, we will restrict our attention now to genera $g > 1$.)

$\Leftrightarrow \beta$ is a normal covering

$\Leftrightarrow X$ is quasiplatonic, has "many automorphisms"

$\Leftrightarrow$ its covering group Γ is a normal subgroup of some triangle group Δ.

Here we have (E.Girondo/J.Wolfart 2005)

Theorem

Different regular dessins on the same quasiplatonic curve are induced by renormalization and inclusions between triangle groups.

Regular dessins

Restriction to more "comfortable" classes of dessins: a dessin D is called regular if there is an automorphism group acting transitively on the edges and preserving incidence, orientation and colours. Automatically, it acts as an automorphism group of the algebraic curve X as well. (For simplicity, we will restrict our attention now to genera $g > 1$.)

$\Leftrightarrow \beta$ is a normal covering

$\Leftrightarrow X$ is quasiplatonic, has "many automorphisms"

$\Leftrightarrow$ its covering group Γ is a normal subgroup of some triangle group Δ.

Here we have (E.Girondo/J.Wolfart 2005)

Theorem

Different regular dessins on the same quasiplatonic curve are induced by renormalization and inclusions between triangle groups.

Regular dessins

Restriction to more "comfortable" classes of dessins: a dessin D is called regular if there is an automorphism group acting transitively on the edges and preserving incidence, orientation and colours. Automatically, it acts as an automorphism group of the algebraic curve X as well. (For simplicity, we will restrict our attention now to genera $g > 1$.)

$\Leftrightarrow$ β is a normal covering

$\Leftrightarrow$ X is quasiplatonic, has "many automorphisms"

$\Leftrightarrow$ its covering group Γ is a normal subgroup of some triangle group Δ.

Here we have (E.Girondo/J.Wolfart 2005)

Theorem

Different regular dessins on the same quasiplatonic curve are induced by renormalization and inclusions between triangle groups.

Uniform dessins

have the same valency in all white vertices, also in all black vertices, and in all faces. The covering groups of their Riemann surfaces are torsion free subgroups of triangle groups Δ. Regular dessins are uniform, but not conversely:

In genus 2 there are 11 regular dessins on 3 non–isomorphic curves (Bolza $\sim$1900), but 579 uniform dessins on ca. 200 (?) curves (Singerman/Syddall 2003). On genus 4 curves there are already more than 14 millions non–isomorphic uniform dessins (Zvonkin).

Can it happen that on one curve X live several uniform dessins of the same type (i.e. with the same valencies)? $\Leftrightarrow$
Are there surface groups contained in several copies of the same triangle group?

Uniform dessins

have the same valency in all white vertices, also in all black vertices, and in all faces. The covering groups of their Riemann surfaces are torsion free subgroups of triangle groups Δ. Regular dessins are uniform, but not conversely:

In genus 2 there are 11 regular dessins on 3 non–isomorphic curves (Bolza $\sim$1900), but 579 uniform dessins on ca. 200 (?) curves (Singerman/Syddall 2003). On genus 4 curves there are already more than 14 millions non–isomorphic uniform dessins (Zvonkin).

Can it happen that on one curve X live several uniform dessins of the same type (i.e. with the same valencies)? $\Leftrightarrow$
Are there surface groups contained in several copies of the same triangle group?

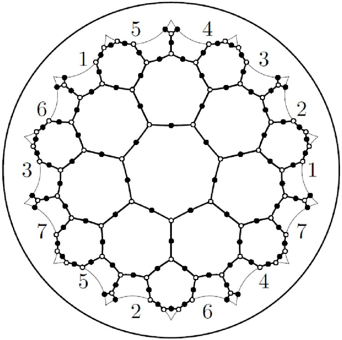
Uniform dessins

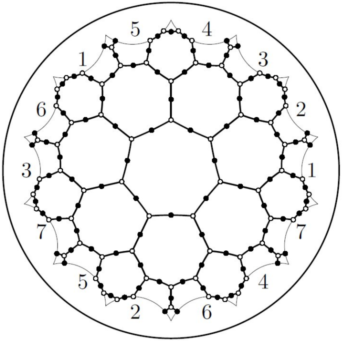
have the same valency in all white vertices, also in all black vertices, and in all faces. The covering groups of their Riemann surfaces are torsion free subgroups of triangle groups Δ. Regular dessins are uniform, but not conversely:

In genus 2 there are 11 regular dessins on 3 non–isomorphic curves (Bolza $\sim$1900), but 579 uniform dessins on ca. 200 (?) curves (Singerman/Syddall 2003). On genus 4 curves there are already more than 14 millions non–isomorphic uniform dessins (Zvonkin).

Can it happen that on one curve X live several uniform dessins of the same type (i.e. with the same valencies)? $\Leftrightarrow$
Are there surface groups contained in several copies of the same triangle group?

Klein's quartic (III)





The regular and one of eight non–regular uniform dessins on Klein's quartic Q, both of type $(2, 3, 7)$ (Syddall 1997, unpublished PhD thesis). How can this happen?

Margulis' obstruction

It means that the surface group Γ of Klein's quartic is contained in a $\Delta = \Delta(2,3,7)$ as normal subgroup, but moreover in 8 copies of Δ as a non–normal subgroup.

A deep result by Margulis implies

Theorem

A non–arithmetic Fuchsian group Γ is contained in a **unique** *maximal Fuchsian group.*

In fact, $\Delta = \Delta(2,3,7)$ and hence the surface group Γ of Klein's quartic are arithmetic Fuchsian groups, therefore Margulis' obstruction does not apply here because $Q = \Gamma\backslash\mathbf{H}$ is a Shimura curve.

Margulis' obstruction

It means that the surface group Γ of Klein's quartic is contained in a $\Delta = \Delta(2,3,7)$ as normal subgroup, but moreover in 8 copies of Δ as a non–normal subgroup.

A deep result by Margulis implies

Theorem

A non–arithmetic Fuchsian group Γ *is contained in a* **unique** *maximal Fuchsian group.*

In fact, $\Delta = \Delta(2,3,7)$ and hence the surface group Γ of Klein's quartic are arithmetic Fuchsian groups, therefore Margulis' obstruction does not apply here because $Q = \Gamma\backslash\mathbf{H}$ is a Shimura curve.

A crashcourse in arithmetic Fuchsian groups

Extremely short survey about an important tool: let A be a

- quaternion algebra
- whose center k is a totally real number field,
- having only one archimedean completion of type $M_2\mathbf{R}$.

In this completion, the (?) maximal order $\mathcal{O}$ of A becomes an $\mathcal{O}_k$–subalgebra of $M_2\mathcal{O}_L$ where $\mathcal{O}_L$ denotes the ring of integers in an at most quadratic extension L of k .

Its unit group $\mathcal{O}^*$ becomes a subgroup of $\mathrm{GL}_2\mathcal{O}_L$, and its elements of *reduced norm* (= determinant) 1 form the norm one group $\mathcal{O}_1^*$. This group acts on the upper half plane as a Fuchsian group of the first kind.

A crashcourse in arithmetic Fuchsian groups

Extremely short survey about an important tool: let A be a

- quaternion algebra
- whose center k is a totally real number field,
- having only one archimedean completion of type $M_2\mathbf{R}$.

In this completion, the (?) maximal order $\mathcal{O}$ of A becomes an $\mathcal{O}_k$–subalgebra of $M_2\mathcal{O}_L$ where $\mathcal{O}_L$ denotes the ring of integers in an at most quadratic extension L of k .

Its unit group $\mathcal{O}^*$ becomes a subgroup of $\mathrm{GL}_2\mathcal{O}_L$, and its elements of *reduced norm* ($=$ determinant) 1 form the norm one group $\mathcal{O}_1^*$. This group acts on the upper half plane as a Fuchsian group of the first kind.

The role of conjugations

$\Gamma < \mathrm{PSL}_2\mathbf{R}$ is called arithmetic, if it (more precisely: its preimage in $\mathrm{SL}_2\mathbf{R}$) is commensurable to such a norm one group. In fact, the triangle group $\Delta(2,3,7)$ is itself a norm one group $\mathcal{O}_1^*$ of a quaternion algebra A with center $k = \mathbf{Q}(\cos\frac{2\pi}{7})$, the cubic real subfield of the cyclotomic field $\mathbf{Q}(\zeta_7)$ of the seventh roots of unity.

So if $\mathcal{O}_1^* > \Gamma < \rho\mathcal{O}_1^*\rho^{-1}$, we may extend ρ to a conjugation of $\mathcal{O}$ and A. Skolem/Noether $\Rightarrow \rho \in A$, w.l.o.g. even $\in \mathcal{O}$.

Then, Γ is contained in the intersection of at least two maximal orders of A, in some Eichler order of A. Conversely, if Γ is contained in Eichler orders, it is contained in several copies of $\mathcal{O}$, hence in our special situation in several copies of $\Delta(2,3,7)$.

The role of conjugations

$\Gamma < \mathrm{PSL}_2\mathbf{R}$ is called arithmetic, if it (more precisely: its preimage in $\mathrm{SL}_2\mathbf{R}$) is commensurable to such a norm one group. In fact, the triangle group $\Delta(2,3,7)$ is itself a norm one group $\mathcal{O}_1^*$ of a quaternion algebra A with center $k = \mathbf{Q}(\cos\frac{2\pi}{7})$, the cubic real subfield of the cyclotomic field $\mathbf{Q}(\zeta_7)$ of the seventh roots of unity.

So if $\mathcal{O}_1^* > \Gamma < \rho\mathcal{O}_1^*\rho^{-1}$, we may extend ρ to a conjugation of $\mathcal{O}$ and A. Skolem/Noether $\Rightarrow \rho \in A$, w.l.o.g. even $\in \mathcal{O}$.

Then, Γ is contained in the intersection of at least two maximal orders of A, in some Eichler order of A. Conversely, if Γ is contained in Eichler orders, it is contained in several copies of $\mathcal{O}$, hence in our special situation in several copies of $\Delta(2,3,7)$.

Main condition

Theorem

Let X be a Shimura curve with surface group $\Gamma \subset \Delta = \Delta(p,q,r)$ where $\Delta = \mathcal{O}_1^*$ is the norm one group of some quaternion algebra A with center k. Then we have several uniform dessins on X of type (p,q,r) if and only if Γ is contained in a group conjugate to a Hecke type congruence subgroup $\Delta_0(\wp)$ for $\wp$ a prime in k not dividing the discriminant $D(A)$.

(For the 85 arithmetic triangle groups, all k have class number 1 (Takeuchi) so we can speak of primes instead of prime ideals.) Recall that

$$\Delta_0(\wp) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta \quad \text{with} \quad c \equiv 0 \bmod \wp \right\}$$

is an intersection $\Delta \cap \rho\Delta\rho^{-1}$ where ρ may be assumed to be an element of $\mathcal{O}$ of *minimal nontrivial norm* such as the matrix $\begin{pmatrix} \wp & 0 \\ 0 & 1 \end{pmatrix}$.

Main condition

Theorem

Let X be a Shimura curve with surface group $\Gamma \subset \Delta = \Delta(p,q,r)$ where $\Delta = \mathcal{O}_1^*$ is the norm one group of some quaternion algebra A with center k. Then we have several uniform dessins on X of type (p,q,r) if and only if Γ is contained in a group conjugate to a Hecke type congruence subgroup $\Delta_0(\wp)$ for $\wp$ a prime in k not dividing the discriminant $D(A)$.

(For the 85 arithmetic triangle groups, all k have class number 1 (Takeuchi) so we can speak of primes instead of prime ideals.) Recall that

$$\Delta_0(\wp) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta \quad \text{with} \quad c \equiv 0 \bmod \wp \right\}$$

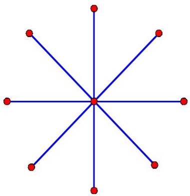
is an intersection $\Delta \cap \rho\Delta\rho^{-1}$ where ρ may be assumed to be an element of $\mathcal{O}$ of *minimal nontrivial norm* such as the matrix $\begin{pmatrix} \wp & 0 \\ 0 & 1 \end{pmatrix}$.

Principal congruence subgroups

$$\Delta(\wp) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta \quad \text{with} \quad c \equiv b \equiv 0 \bmod \wp\,,\ a \equiv d \equiv 1 \bmod \wp \right\}$$

of prime level $\wp$ is not only contained in one such intersection $\Delta_0(\wp) = \Delta \cap \rho\Delta\rho^{-1}$ but in $q+1$ groups of this type, all conjugate in Δ, where q is the norm of $\wp$.

Example: In the case of Klein's quartic, $\Gamma = \Delta(\wp) \lhd \Delta(2,3,7)$ for the prime $\wp = 2 - 2\cos\frac{2\pi}{7}$ of norm $q = 7$ in the center field $k = \mathbf{Q}(\cos\frac{2\pi}{7})$. We may symbolize the different dessins of type $(2,3,7)$ (or the different maximal groups above $\Gamma = \Delta(\wp)$) by the vertices of the following graph:

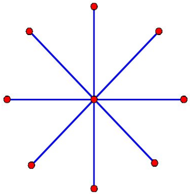


Principal congruence subgroups

$$\Delta(\wp) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta \quad \text{with} \quad c \equiv b \equiv 0 \bmod \wp\,,\ a \equiv d \equiv 1 \bmod \wp \right\}$$

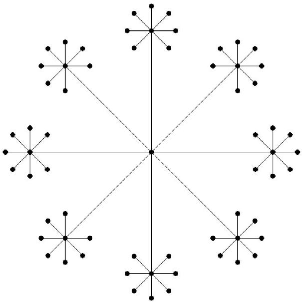
of prime level $\wp$ is not only contained in one such intersection $\Delta_0(\wp) = \Delta \cap \rho\Delta\rho^{-1}$ but in $q+1$ groups of this type, all conjugate in Δ, where q is the norm of $\wp$.

Example: In the case of Klein's quartic, $\Gamma = \Delta(\wp) \lhd \Delta(2,3,7)$ for the prime $\wp = 2 - 2\cos\frac{2\pi}{7}$ of norm $q = 7$ in the center field $k = \mathbf{Q}(\cos\frac{2\pi}{7})$. We may symbolize the different dessins of type $(2,3,7)$ (or the different maximal groups above $\Gamma = \Delta(\wp)$) by the vertices of the following graph:



Higher level

principal congruence subgroups like $\Delta(\wp^n)$ are contained in $1+(q+1)+(q+1)q+\ldots+(q+1)q^{n-1}$ different copies of Δ, for $q=7$, $n=2$ visualised by the vertices of



and so on: for prime power levels one gets always a finite subtree of a Bruhat–Tits tree (Bass–Serre tree? Brandt tree?).

Fields of moduli and fields of definition

Recall that smooth complex projective algebraic curves C with Belyĭ functions β, i.e. with dessins, can be defined by algebraic equations with coefficients in some number field K. This is a field of definition for C, and we may introduce in the same way a common field of definition for C and β, a *field of definition for the dessin.*

These fields of definition are not unique, but all contain the field of moduli $M(C)$ of the curve, defined as the common fixed field of all $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$ with the property that there is an isomorphism $f_\sigma : C \to C^\sigma$.
Exercise: this moduli field depends only on the isomorphism class of C.

Similarly, we consider *moduli fields of dessins* $M(C,\beta)$ requiring from the isomorphisms f_σ the additional compatibility condition

$$\beta^\sigma \circ f_\sigma = \beta \quad .$$

Apparently, $M(C) \subset M(C,\beta)$.

Fields of moduli and fields of definition

Recall that smooth complex projective algebraic curves C with Belyĭ functions β, i.e. with dessins, can be defined by algebraic equations with coefficients in some number field K. This is a field of definition for C, and we may introduce in the same way a common field of definition for C and β, a *field of definition for the dessin.*

These fields of definition are not unique, but all contain the field of moduli $M(C)$ of the curve, defined as the common fixed field of all $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$ with the property that there is an isomorphism $f_\sigma : C \to C^\sigma$.
Exercise: this moduli field depends only on the isomorphism class of C.

Similarly, we consider *moduli fields of dessins* $M(C, \beta)$ requiring from the isomorphisms f_σ the additional compatibility condition

$$\beta^\sigma \circ f_\sigma = \beta \quad .$$

Apparently, $M(C) \subset M(C, \beta)$.

Fields of moduli and fields of definition

Recall that smooth complex projective algebraic curves C with Belyĭ functions β, i.e. with dessins, can be defined by algebraic equations with coefficients in some number field K. This is a field of definition for C, and we may introduce in the same way a common field of definition for C and β, a *field of definition for the dessin.*

These fields of definition are not unique, but all contain the field of moduli $M(C)$ of the curve, defined as the common fixed field of all $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$ with the property that there is an isomorphism $f_\sigma : C \to C^\sigma$.
Exercise: this moduli field depends only on the isomorphism class of C.

Similarly, we consider *moduli fields of dessins* $M(C, \beta)$ requiring from the isomorphisms f_σ the additional compatibility condition

$$\beta^\sigma \circ f_\sigma = \beta \quad .$$

Apparently, $M(C) \subset M(C, \beta)$.

Weil's cocycle condition

In general, it is much easier to determine fields of moduli than fields of definition.

Can C (or (C,β) or $(C,\mathrm{Aut}\, C)$) be defined over its field of moduli?

Theorem (A. Weil)

Yes, if and only if for all $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/M(C)$ *there are isomorphisms* $f_\sigma : C \to C^\sigma$ *such that for all* σ , τ

$$f_{\tau\sigma} = f_\sigma^\tau \circ f_\tau \quad .$$

This criterion is extremely useful in particular in rigid situations, i.e. if only one isomorphism $f_\sigma : C \to C^\sigma$ exists:

If C has no nontrivial automorphism, it can be defined over $M(C)$.

Weil's cocycle condition

In general, it is much easier to determine fields of moduli than fields of definition.

Can C (or (C, β) or $(C, \mathrm{Aut}\, C)$) be defined over its field of moduli?

Theorem (A. Weil)

Yes, if and only if for all $\sigma \in \mathrm{Gal}\, \overline{\mathbf{Q}}/M(C)$ there are isomorphisms $f_\sigma : C \to C^\sigma$ such that for all σ , τ

$$f_{\tau\sigma} = f_\sigma^\tau \circ f_\tau \quad .$$

This criterion is extremely useful in particular in rigid situations, i.e. if only one isomorphism $f_\sigma : C \to C^\sigma$ exists:

If C has no nontrivial automorphism, it can be defined over $M(C)$.

Weil's cocycle condition

In general, it is much easier to determine fields of moduli than fields of definition.

Can C (or (C,β) or $(C,\mathrm{Aut}\,C)$) be defined over its field of moduli?

Theorem (A. Weil)

Yes, if and only if for all $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/M(C)$ there are isomorphisms $f_\sigma : C \to C^\sigma$ such that for all σ, τ

$$f_{\tau\sigma} = f_\sigma^\tau \circ f_\tau \quad .$$

This criterion is extremely useful in particular in rigid situations, i.e. if only one isomorphism $f_\sigma : C \to C^\sigma$ exists:

If C has no nontrivial automorphism, it can be defined over $M(C)$.

Rigidify !

Coombes and Harbater invented a method how to apply rigidity also in situations which are a priori non–rigid.

Theorem

Regular dessins can be defined over their moduli fields $M(C,\beta)$.

Idea: restrict the choice of the isomorphisms f_σ by imposing even more conditions: $f_\sigma(x) = \sigma(x) \in C^\sigma$ for some preimage $x = \beta^{-1}(r) \in C$ of a rational $r \neq 0, 1, \infty$. In other words: consider even triplets (C, β, x) instead!

Consequence (Wolfart '97/'06):

Theorem

Quasiplatonic curves C can be defined over their fields of moduli $M(C)$.

Rigidify !

Coombes and Harbater invented a method how to apply rigidity also in situations which are a priori non–rigid.

Theorem

Regular dessins can be defined over their moduli fields $M(C, \beta)$ *.*

Idea: restrict the choice of the isomorphisms f_σ by imposing even more conditions: $f_\sigma(x) = \sigma(x) \in C^\sigma$ for some preimage $x = \beta^{-1}(r) \in C$ of a rational $r \neq 0, 1, \infty$. In other words: consider even triplets (C, β, x) instead!

Consequence (Wolfart '97/'06):

Theorem

Quasiplatonic curves C *can be defined over their fields of moduli* $M(C)$ *.*

Some explicit results

In particular, all quasiplatonic curves in genera $1 < g < 6$ can be defined over $\mathbf{Q}$ since they are uniquely determined by type and automorphism groups of their regular dessins.

What happens for the Shimura curves $S(\wp^n) := \Delta(\wp^n)\backslash\mathbf{H}$ discussed above where Δ is a (norm one) arithmetic triangle group and $\wp$ a prime in the trace field not dividing the discriminant of the quaternion algebra?

- $M(S(\wp)) = \mathbf{Q}$ for $\Delta(2,3,7)$ if $\wp = p \equiv \pm 2, \pm 3 \bmod 7$
  $M(S(\wp)) = k = \mathbf{Q}(\cos\frac{2\pi}{7})$ if $\wp \mid p \equiv \pm 1 \bmod 7$ (Streit 2000, Džambić 2007),
- $M(S(\wp))$ is the splitting field of $\wp$ in the trace field k of Δ (Feierabend 2008, Clark/Voight 2011, preprint).
- For all $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$ one has $S(\wp)^\sigma = S(\sigma(\wp))$.

Some explicit results

In particular, all quasiplatonic curves in genera $1 < g < 6$ can be defined over $\mathbf{Q}$ since they are uniquely determined by type and automorphism groups of their regular dessins.

What happens for the Shimura curves $S(\wp^n) := \Delta(\wp^n)\backslash\mathbf{H}$ discussed above where Δ is a (norm one) arithmetic triangle group and $\wp$ a prime in the trace field not dividing the discriminant of the quaternion algebra?

- $M(S(\wp)) = \mathbf{Q}$ for $\Delta(2,3,7)$ if $\wp = p \equiv \pm 2, \pm 3 \bmod 7$
  $M(S(\wp)) = k = \mathbf{Q}(\cos\frac{2\pi}{7})$ if $\wp \mid p \equiv \pm 1 \bmod 7$ (Streit 2000, Džambić 2007),
- $M(S(\wp))$ is the splitting field of $\wp$ in the trace field k of Δ (Feierabend 2008, Clark/Voight 2011, preprint).
- For all $\sigma \in \mathrm{Gal}\,\overline{\mathbf{Q}}/\mathbf{Q}$ one has $S(\wp)^\sigma = S(\sigma(\wp))$.

Latest news

Theorem

Under the same hypotheses we have $M(S(\wp^n)) = M(S(\wp))$ *.*

(Recall: all curves quasiplatonic, hence *moduli fields = fields of definition.*)
Reason. All Galois conjugates of $S(\wp^n)$ have the same pattern of uniform dessins coming from Δ . This pattern (described via the finite subtree of the Bruhat–Tits tree) occurs only for curves coming from congruence subgroups of Δ for prime power levels Galois conjugate to $\wp^n$. Therefore, the only nontrivial Galois actions are of the type $S(\wp)^\sigma = S(\sigma(\wp))$.
(J. Voight: arguments using moduli spaces are also available.)

What happens with the moduli field of the uniform non–regular dessins on $S(\wp^n)$? Are they also fields of definition? And which ones?

Latest news

Theorem

Under the same hypotheses we have $M(S(\wp^n)) = M(S(\wp))$.

(Recall: all curves quasiplatonic, hence *moduli fields* = *fields of definition.*)
Reason. All Galois conjugates of $S(\wp^n)$ have the same pattern of uniform dessins coming from Δ . This pattern (described via the finite subtree of the Bruhat–Tits tree) occurs only for curves coming from congruence subgroups of Δ for prime power levels Galois conjugate to $\wp^n$. Therefore, the only nontrivial Galois actions are of the type $S(\wp)^\sigma = S(\sigma(\wp))$.
(J. Voight: arguments using moduli spaces are also available.)

What happens with the moduli field of the uniform non–regular dessins on $S(\wp^n)$? Are they also fields of definition? And which ones?

Uniform dessins: fields of moduli

As always, let $S(\wp^n) := \Delta(\wp^n)\backslash\mathbf{H}$ be the Shimura curve uniformised by the principal congruence subgroup $\Delta(\wp^n)$ of a (maximal) arithmetic triangle group Δ where $\wp$ is a prime in the center field of the quaternion algebra A not dividing the discriminant of A.

Theorem

For all Belyĭ functions β induced by a copy of Δ on $S(\wp^n)$ the moduli field of the uniform dessin is

$$M(S(\wp^n),\beta) = M(S(\wp^n)) = M(S(\wp)) \quad .$$

Idea of proof: We may suppose that $S(\wp)$ is defined over $M(S(\wp))$. Then all β^σ are Belyĭ functions on $S(\wp^n)$ of the same type as β and are of the shape $\beta\circ\alpha$ for an automorphism $\alpha = \alpha_\sigma$ of $S(\wp^n)$.

Uniform dessins: fields of definition

Theorem

Under the same hypotheses, $M(S(\wp))$ is even a field of definition of $(S(\wp^n), \beta)$.

Idea: The automorphism group $G = G(\beta)$ of the dessin, i.e. of $(S(\wp^n), \beta)$, is always conjugate to some $\Delta_0(\wp^m)/\Delta(\wp^n)$ for some integer $0 \le m \le n$. For a given m , the possible subgroups $G \subset \mathrm{Aut}\, S(\wp^n)$ are in 1–to–1 correspondence to the possible β with isomorphic automorphism groups. So,

$$G^\sigma \;=\; G(\beta^\sigma)$$

$= \alpha_\sigma^{-1} G \alpha_\sigma$ for some $\alpha_\sigma \in \mathrm{Aut}\, S(\wp^n)$ which can be made unique using a variant of the Coombes–Harbater trick.

Uniform dessins: fields of definition

Theorem

Under the same hypotheses, $M(S(\wp))$ *is even a field of definition of* $(S(\wp^n), \beta)$ *.*

Idea: The automorphism group $G = G(\beta)$ of the dessin, i.e. of $(S(\wp^n), \beta)$, is always conjugate to some $\Delta_0(\wp^m)/\Delta(\wp^n)$ for some integer $0 \le m \le n$. For a given m, the possible subgroups $G \subset \mathrm{Aut}\, S(\wp^n)$ are in 1–to–1 correspondence to the possible β with isomorphic automorphism groups. So,

$$G^\sigma \ = \ G(\beta^\sigma)$$

$= \ \alpha_\sigma^{-1} G \alpha_\sigma$ for some $\alpha_\sigma \in \mathrm{Aut}\, S(\wp^n)$ which can be made unique using a variant of the Coombes–Harbater trick.

Simultaneous field of definition

That the moduli field $M(S(\wp))$ is a field of definition for all $(S(\wp^n), \beta)$ does not mean that there is a model for $S(\wp^n)$ defined over $M(S(\wp))$ in which all uniform β are given simultaneously as rational functions with coefficients in $M(S(\wp))$. How could such a common field of definition look like?

An easy argument using Galois theory for function fields gives

Theorem

All uniform Belyĭ functions β on $S(\wp^n)$ can be defined simultaneaously over a common field of definition for $S(\wp^n)$ and $\mathrm{Aut}\, S(\wp^n)$.

Simultaneous field of definition

That the moduli field $M(S(\wp))$ is a field of definition for all $(S(\wp^n), \beta)$ does not mean that there is a model for $S(\wp^n)$ defined over $M(S(\wp))$ in which all uniform β are given simultaneously as rational functions with coefficients in $M(S(\wp))$. How could such a common field of definition look like?

An easy argument using Galois theory for function fields gives

Theorem

All uniform Belyĭ functions β on $S(\wp^n)$ can be defined simultaneaously over a common field of definition for $S(\wp^n)$ and $\mathrm{Aut}\, S(\wp^n)$.

The moduli field of the automorphism group

Galois cohomology (Clark/Voight) or another application of the Coombes–Harbater trick shows

Theorem

The pair $(S(\wp^n), \mathrm{Aut}\, S(\wp^n))$ *can be defined over* $M(S(\wp^n), \mathrm{Aut}\, S(\wp^n))$ *.*

Clark and Voight can show that $M(S(\wp), \mathrm{Aut}\, S(\wp))$ is an extension of small degree of $M(S(\wp))$: In the case of Klein's quartic, this common field minimal of definition is $\mathbf{Q}(\sqrt{-7})$.

In higher levels, this is no longer true!

Thank you for your attention – and merry Christmas!

The moduli field of the automorphism group

Galois cohomology (Clark/Voight) or another application of the Coombes–Harbater trick shows

Theorem

The pair $(S(\wp^n), \mathrm{Aut}\, S(\wp^n))$ *can be defined over* $M(S(\wp^n), \mathrm{Aut}\, S(\wp^n))$.

Clark and Voight can show that $M(S(\wp), \mathrm{Aut}\, S(\wp))$ is an extension of small degree of $M(S(\wp))$: In the case of Klein's quartic, this common field minimal of definition is $\mathbf{Q}(\sqrt{-7})$.

In higher levels, this is no longer true!

Thank you for your attention – and merry Christmas!