

Polygonal billiards and homogeneous foliations

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1 Polygonal Billiards

2 Homogeneous foliations on \mathbb{C}^2

3 Dictionary

4 Foliations in $\mathbb{RP}(3)$

Polygonal Billiards

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbb{RP}(3)$

Dynamical systems whose ingredients are:

Polygonal Billiards

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbb{RP}(3)$

Dynamical systems whose ingredients are: a table (Euclidean polygon P)

Polygonal Billiards

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbb{RP}(3)$

Dynamical systems whose ingredients are: a table (Euclidean polygon P) + ball (point particle)

Polygonal Billiards

Polygonal billiards and homogeneous foliations

Ferrán Valdez

Outline

Polygonal Billiards

Homogeneous foliations on \mathbb{C}^2

Dictionary

Foliations in $\mathbb{RP}(3)$

Dynamical systems whose ingredients are: a table (Euclidean polygon P) + ball (point particle) + reflection law (à la Descartes).

Polygonal Billiards

Polygonal billiards and homogeneous foliations

Ferrán Valdez

Outline

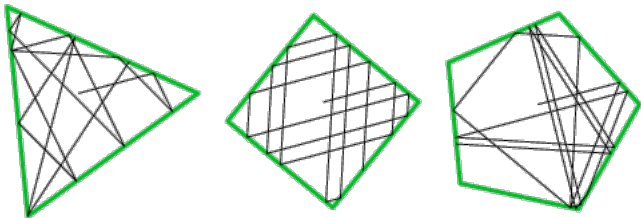
Polygonal Billiards

Homogeneous foliations on \mathbb{C}^2

Dictionary

Foliations in $\mathbb{RP}(3)$

Dynamical systems whose ingredients are: a table (Euclidean polygon P) + ball (point particle) + reflection law (à la Descartes).



Polygonal Billiards

Polygonal billiards and homogeneous foliations

Ferrán Valdez

Outline

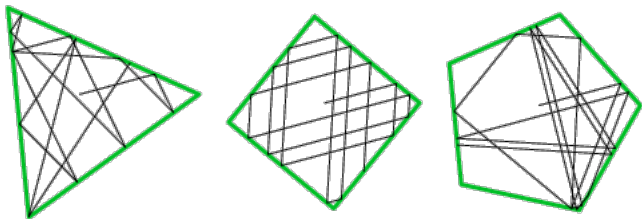
Polygonal Billiards

Homogeneous foliations on \mathbb{C}^2

Dictionary

Foliations in $\mathbb{RP}(3)$

Dynamical systems whose ingredients are: a table (Euclidean polygon P) + ball (point particle) + reflection law (à la Descartes).



(No friction, motion ends if we hit an infinitesimal pocket)

Polygonal billiards and translation surfaces

Polygonal billiards and homogeneous foliations

Ferrán Valdez

Outline

Polygonal Billiards

Homogeneous foliations on \mathbb{C}^2

Dictionary

Foliations in $\mathbb{RP}(3)$

Idea (Fox-Kershner/Katok-Zemljakov): associate to each polygon P a “translation” (or “flat”) surface

$$S_P \rightarrow P$$

on which geodesics project to billiard trajectories.

Polygonal billiards and translation surfaces

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbb{RP}(3)$

Idea (Fox-Kershner/Katok-Zemljakov): associate to each polygon P a “translation” (or “flat”) surface

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Polygonal billiards and translation surfaces

Polygonal billiards and homogeneous foliations

Ferrán Valdez

Outline

Polygonal Billiards

Homogeneous foliations on \mathbb{C}^2

Dictionary

Foliations in $\mathbb{RP}(3)$

Idea (Fox-Kershner/Katok-Zemljakov): associate to each polygon P a “translation” (or “flat”) surface

$$S_P \rightarrow P$$

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Main open questions: *is there a periodic trajectory in every triangular billiard?*

Polygonal billiards and translation surfaces

Polygonal billiards and homogeneous foliations

Ferrán Valdez

Outline

Polygonal Billiards

Homogeneous foliations on \mathbb{C}^2

Dictionary

Foliations in $\mathbb{RP}(3)$

Idea (Fox-Kershner/Katok-Zemljakov): associate to each polygon P a “translation” (or “flat”) surface

$$S_P \rightarrow P$$

on which geodesics project to billiard trajectories. Studying the billiard is equivalent to studying the geodesic flow on S_P .

Main open questions: *is there a periodic trajectory in every triangular billiard? is the geodesic flow on the generic S_P recurrent or dissipative?*

Homogeneous holomorphic foliations on \mathbf{C}^2

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbf{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

Homogeneous = invariant under the homothety action of \mathbf{C}^* on \mathbf{C}^2 .

Homogeneous holomorphic foliations on \mathbf{C}^2

Homogeneous = invariant under the homothety action of \mathbf{C}^* on \mathbf{C}^2 . We will consider foliations \mathcal{F}_λ defined by the holomorphic 1-form ω_λ

$$\frac{\omega_\lambda}{z_1 z_2 (z_2 - z_1)} = \lambda_1 \frac{dz_1}{z_1} + \lambda_2 \frac{dz_2}{z_2} + \lambda_3 \frac{d(z_2 - z_1)}{z_2 - z_1},$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\pi\lambda_i$ are the interior angles of a triangle P .

Homogeneous holomorphic foliations on \mathbf{C}^2

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where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\pi\lambda_i$ are the interior angles of a triangle P . In the dual picture, \mathcal{F}_λ is given by the integral curves of the vector field X_λ

$$(\lambda_2 z_1 (z_2 - z_1) + \lambda_3 z_1 z_2) \frac{\partial}{\partial z_1} + (\lambda_1 z_2 (z_1 - z_2) + \lambda_3 z_1 z_2) \frac{\partial}{\partial z_2}$$

Homogeneous holomorphic foliations on \mathbb{C}^2

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbb{RP}(3)$

Using E. Paul's work we can assure that the fibers of

$$F_\lambda(z_1, z_2) = z_1^{\lambda_1} z_2^{\lambda_2} (z_2 - z_1)^{\lambda_3}$$

over \mathbb{C}^* are the leaves of \mathcal{F}_λ in $\mathbb{C}^2 \setminus \{z_1 z_2 (z_2 - z_1) = 0\}$. In other words, F_λ is a *first integral* for \mathcal{F}_λ .

Homogeneous holomorphic foliations on \mathbf{C}^2

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbf{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

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over \mathbf{C}^* are the leaves of \mathcal{F}_λ in $\mathbf{C}^2 \setminus \{z_1 z_2 (z_2 - z_1) = 0\}$. In other words, F_λ is a *first integral* for \mathcal{F}_λ . Hence, there are 3 types of leaves : those in $F_\lambda^{-1}(0)$ (the tangent cone) and $F_\lambda^{-1}(t)$ with $t \in \mathbf{C}^*$ (generic leaf, denoted L).

Flat surface structure of leaves

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbb{RP}(3)$

Consider the 1-form η on \mathbb{C}^2 satisfying $\eta(X_\lambda) = 1$.

Flat surface structure of leaves

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbb{RP}(3)$

Consider the 1-form η on \mathbb{C}^2 satisfying $\eta(X_\lambda) = 1$. Then, for every $L \in \mathcal{F}_\lambda$, $\eta|_L$ is zero free. We can then define new coordinates in L by the map:

$$z(p) = \int_{p_0}^p \eta|_L$$

Flat surface structure of leaves

Consider the 1-form η on \mathbb{C}^2 satisfying $\eta(X_\lambda) = 1$. Then, for every $L \in \mathcal{F}_\lambda$, $\eta|_L$ is zero free. We can then define new coordinates in L by the map:

$$z(p) = \int_{p_0}^p \eta|_L$$

Remark that, if we change base points in some small patch, then our new coord. change by a translation:

$$c := \int_{p_0}^p \eta|_L - \int_{p_1}^p \eta|_L = \int_{p_0}^{p_1} \eta|_L$$

Flat surface structure of leaves

Consider the 1-form η on \mathbb{C}^2 satisfying $\eta(X_\lambda) = 1$. Then, for every $L \in \mathcal{F}_\lambda$, $\eta|_L$ is zero free. We can then define new coordinates in L by the map:

$$z(p) = \int_{p_0}^p \eta|_L$$

Remark that, if we change base points in some small patch, then our new coord. change by a translation:

$$c := \int_{p_0}^p \eta|_L - \int_{p_1}^p \eta|_L = \int_{p_0}^{p_1} \eta|_L$$

In other words, the leaves of \mathcal{F}_λ have a “natural” translation surface structure. Notation $(L, \eta|_L)$

Dictionary

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbf{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

Write $X_\lambda = X_1 + iX_2$ and define \mathcal{F}_0 to be the (real) foliation on \mathbf{C}^2 defined by the integral curves of X_1 .

Dictionary

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbf{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

Write $X_\lambda = X_1 + iX_2$ and define \mathcal{F}_0 to be the (real) foliation on \mathbf{C}^2 defined by the integral curves of X_1 . The restriction of \mathcal{F}_0 to a generic leaf $L \in \mathcal{F}$ is a foliation.

Dictionary

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbf{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

Write $X_\lambda = X_1 + iX_2$ and define \mathcal{F}_0 to be the (real) foliation on \mathbf{C}^2 defined by the integral curves of X_1 . The restriction of \mathcal{F}_0 to a generic leaf $L \in \mathcal{F}$ is a foliation. Locally, the map $z : L \rightarrow (L, \eta_1)$ sends \mathcal{F}_0 to the set of integral curves of $\frac{\partial}{\partial t}$.

Dictionary

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbf{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

Write $X_\lambda = X_1 + iX_2$ and define \mathcal{F}_0 to be the (real) foliation on \mathbf{C}^2 defined by the integral curves of X_1 . The restriction of \mathcal{F}_0 to a generic leaf $L \in \mathcal{F}$ is a foliation. Locally, the map $z : L \rightarrow (L, \eta_1)$ sends \mathcal{F}_0 to the set of integral curves of $\frac{\partial}{\partial t}$.

Theorem.

Let $L \in \mathcal{F}_\lambda$ be a fixed generic leaf. Then,

Dictionary

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbf{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

Write $X_\lambda = X_1 + iX_2$ and define \mathcal{F}_0 to be the (real) foliation on \mathbf{C}^2 defined by the integral curves of X_1 . The restriction of \mathcal{F}_0 to a generic leaf $L \in \mathcal{F}$ is a foliation. Locally, the map $z : L \rightarrow (L, \eta_1)$ sends \mathcal{F}_0 to the set of integral curves of $\frac{\partial}{\partial t}$.

Theorem.

Let $L \in \mathcal{F}_\lambda$ be a fixed generic leaf. Then,

- 1 There exists a translation surface isomorphism $\phi : (L, \eta_1) \rightarrow S_P$.

Dictionary

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbb{RP}(3)$

Write $X_\lambda = X_1 + iX_2$ and define \mathcal{F}_0 to be the (real) foliation on \mathbb{C}^2 defined by the integral curves of X_1 . The restriction of \mathcal{F}_0 to a generic leaf $L \in \mathcal{F}$ is a foliation. Locally, the map $z : L \rightarrow (L, \eta_1)$ sends \mathcal{F}_0 to the set of integral curves of $\frac{\partial}{\partial t}$.

Theorem.

Let $L \in \mathcal{F}_\lambda$ be a fixed generic leaf. Then,

- 1 There exists a translation surface isomorphism $\phi : (L, \eta_1) \rightarrow S_\rho$. In particular, the foliation (L, \mathcal{F}_0) is conjugated by ϕ to a foliation on S_ρ formed by geodesics parallel to some direction θ .

Dictionary

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbf{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

Write $X_\lambda = X_1 + iX_2$ and define \mathcal{F}_0 to be the (real) foliation on \mathbf{C}^2 defined by the integral curves of X_1 . The restriction of \mathcal{F}_0 to a generic leaf $L \in \mathcal{F}$ is a foliation. Locally, the map $z : L \rightarrow (L, \eta_1)$ sends \mathcal{F}_0 to the set of integral curves of $\frac{\partial}{\partial t}$.

Theorem.

Let $L \in \mathcal{F}_\lambda$ be a fixed generic leaf. Then,

- 1 There exists a translation surface isomorphism $\phi : (L, \eta_1) \rightarrow S_\rho$. In particular, the foliation (L, \mathcal{F}_0) is conjugated by ϕ to a foliation on S_ρ formed by geodesics parallel to some direction θ .
- 2 If we change L for $\rho e^{i\alpha} L$, then \mathcal{F}_0 restricted to $\rho e^{i\alpha} L$ is conjugated to the foliation on S_ρ formed by geodesics parallel to $\theta + \alpha$.

Sketch of proof.

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbb{RP}(3)$

Aim: construct $\phi : (L, \eta_1) \longrightarrow S_P$.

Sketch of proof.

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbb{RP}(3)$

Aim: construct $\phi : (L, \eta_1) \rightarrow S_P$.

Locally, X_λ restricted to L is of the form

$$t^{1-\lambda_2}(t-1)^{1-\lambda_3}\partial/\partial t, \quad t \in \mathbf{H}$$

Sketch of proof.

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbb{RP}(3)$

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Therefore

$$z(t) := \int^t \xi^{\lambda_2-1}(\xi-1)^{\lambda_3-1} d\xi$$

defines the translation surface structure of L (& locally rectifies X_1).

Sketch of proof.

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbb{RP}(3)$

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defines the translation surface structure of L (& locally rectifies X_1). Remark that f is a Schwarz-Christoffel transformation.

Sketch of proof.

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbb{RP}(3)$

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“Hard” part: show that z extends to define ϕ . □

Remarks

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbb{RP}(3)$

The triangular billiard has a periodic orbit iff the vector field

Remarks

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbb{RP}(3)$

The triangular billiard has a periodic orbit iff the vector field

$$\begin{aligned} 2X_1 = & [\lambda_2(x_1^2 - y_1^2) - (\lambda_2 + \lambda_3)(x_1x_2 - y_1y_2)]\partial/\partial x_1 + \\ & [2\lambda_2x_1y_1 - (\lambda_2 + \lambda_3)(x_1y_2 + x_2y_1)]\partial/\partial y_1 + \\ & [\lambda_1(x_2^2 - y_2^2) - (\lambda_1 + \lambda_3)(x_1x_2 - y_1y_2)]\partial/\partial x_2 + \\ & [2\lambda_1x_2y_2 - (\lambda_1 + \lambda_3)(x_1y_2 + x_2y_1)]\partial/\partial y_2 \end{aligned}$$

has a periodic orbit.

Foliations in $\mathbf{RP}(3)$

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbf{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

The phase space of the billiard is three dimensional!

Foliations in $\mathbf{RP}(3)$

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

The phase space of the billiard is three dimensional! Since everything is homogeneous look at the induced foliations in $\mathbf{RP}(3)$ (or in S^3).

Foliations in $\mathbf{RP}(3)$

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

The phase space of the billiard is three dimensional! Since everything is homogeneous look at the induced foliations in $\mathbf{RP}(3)$ (or in S^3).

Let \mathcal{G}_λ be the foliation defined in homogeneous coordinates by:

$$\alpha := i_{\mathbb{R}} i_{X_1} i_{X_2} (dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2)$$

The singular locus of \mathcal{G}_λ is $\pi(z_1 z_2 (z_2 - z_1) = 0)$

Foliations in $\mathbf{RP}(3)$

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

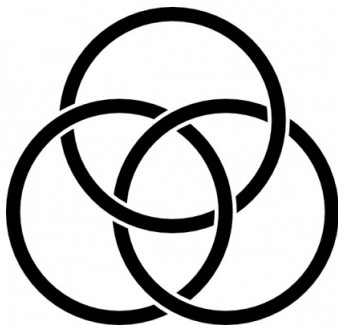
Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$



The Borromean rings “=” $Sing\mathcal{G}_\lambda$.

Foliations in $\mathbf{RP}(3)$

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

Let $L \in \mathcal{F}_\lambda$ be fixed. If there exist $n, m \in \mathbb{Z}$ such that $n\lambda_i + m\lambda_j - \frac{1}{2} \in \mathbb{Z}$, then

$$\pi|_L : L \rightarrow \pi(L)$$

is a $2 : 1$ map.

Foliations in $\mathbf{RP}(3)$

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

Let $L \in \mathcal{F}_\lambda$ be fixed. If there exist $n, m \in \mathbb{Z}$ such that $n\lambda_i + m\lambda_j - \frac{1}{2} \in \mathbb{Z}$, then

$$\pi|_L : L \rightarrow \pi(L)$$

is a $2 : 1$ map. Else, it is a bijection. Hence we can talk about the generic leaf of \mathcal{G}_λ .

Foliations in $\mathbf{RP}(3)$

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

Let $L \in \mathcal{F}_\lambda$ be fixed. If there exist $n, m \in \mathbb{Z}$ such that $n\lambda_i + m\lambda_j - \frac{1}{2} \in \mathbb{Z}$, then

$$\pi|_L : L \rightarrow \pi(L)$$

is a $2 : 1$ map. Else, it is a bijection. Hence we can talk about the generic leaf of \mathcal{G}_λ .

Local picture near $Sing\mathcal{G}_\lambda$: (up to analytic change of coordinates) \mathcal{G}_λ is given by

$$XdY - YdX$$

(Open Book / Kupka phenomenon)

Foliations in $\mathbf{RP}(3)$

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

Global picture near $Sing\mathcal{G}_\lambda$. Let

$$\pi : \widetilde{\mathbf{RP}(3)} \rightarrow \mathbf{RP}(3)$$

be the blow-up along $Sing(\mathcal{G}_\lambda)$ and $\widetilde{\mathcal{G}}_\lambda$ the foliation defined by $\pi^*\alpha$.

Foliations in $\mathbf{RP}(3)$

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

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$$\pi : \widetilde{\mathbf{RP}(3)} \rightarrow \mathbf{RP}(3)$$

be the blow-up along $Sing(\mathcal{G}_\lambda)$ and $\widetilde{\mathcal{G}}_\lambda$ the foliation defined by $\pi^*\alpha$. Each component of the Borromean rings defines a “torus” T_i .

Foliations in $\mathbf{RP}(3)$

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

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$$\pi : \widetilde{\mathbf{RP}(3)} \rightarrow \mathbf{RP}(3)$$

be the blow-up along $Sing(\mathcal{G}_\lambda)$ and $\widetilde{\mathcal{G}}_\lambda$ the foliation defined by $\pi^*\alpha$. Each component of the Borromean rings defines a “torus” T_i . Let (x, y) be local coordinates for T_i , then the trace of a leaf $L \in \widetilde{\mathcal{G}}_\lambda$ in T_i is of the form:

$$\lambda_i x + (1 - \lambda_i) y = k, \quad k \in \mathbb{R}$$

In other words, $\widetilde{\mathcal{G}}_\lambda|_{T_i}$ is defined by “parallel lines”.

Foliations in $\mathbf{RP}(3)$

Polygonal
billiards and
homogeneous
foliations

Ferrán Valdez

Outline

Polygonal
Billiards

Homogeneous
foliations on
 \mathbb{C}^2

Dictionary

Foliations in
 $\mathbf{RP}(3)$

Let \mathcal{G}_0 be the foliation on $\mathbf{RP}(3)$ defined by X_1 .

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Ferrán Valdez

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billiards and
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Billiards

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foliations on
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$$p_1 = [1 : 0 : 0 : 0] \quad p_2 = [0 : 0 : 1 : 0] \quad p_3 = [1 : 0 : 1 : 0]$$

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billiards and
homogeneous
foliations

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Outline

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Billiards

Homogeneous
foliations on
 \mathbb{C}^2

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Local picture near $Sing(\mathcal{G}_0)$. The foliation is conjugated to the foliation defined by

$$\lambda_j(x + x^3) \frac{\partial}{\partial x} + [-y + x(\dots)] \frac{\partial}{\partial y} + [-z + x(\dots)] \frac{\partial}{\partial z}$$

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billiards and
homogeneous
foliations

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Outline

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Billiards

Homogeneous
foliations on
 \mathbb{C}^2

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Remark the invariant (local) manifold $\{x = 0\}$!

¡Muchas gracias a todos y feliz Navidad!

vielen Dank und schöne
Weihnachten!



- Polygonal billiards and homogeneous foliations
- Ferrán Valdez
- Outline
- Polygonal Billiards
- Homogeneous foliations on \mathbb{C}^2
- Dictionary
- Foliations in $RP(3)$