1. Polygonal Billiards

2. Homogeneous foliations on $\mathbb{C}^2$

3. Dictionary

4. Foliations in $\mathbb{RP}(3)$
Dynamical systems whose ingredients are:
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(No friction, motion ends if we hit an infinitesimal pocket)
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Polygonal billiards and translation surfaces

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Main open questions: is there a periodic trajectory in every triangular billiard? is the geodesic flow on the generic $S_P$ recurrent or dissipative?
Homogeneous holomorphic foliations on $\mathbb{C}^2$

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$$\frac{\omega_\lambda}{z_1 z_2(z_2 - z_1)} = \lambda_1 \frac{dz_1}{z_1} + \lambda_2 \frac{dz_2}{z_2} + \lambda_3 \frac{d(z_2 - z_1)}{z_2 - z_1},$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\pi \lambda_i$ are the interior angles of a triangle $P$. 
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where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\pi \lambda_i$ are the interior angles of a triangle $P$. In the dual picture, $\mathcal{F}_\lambda$ is given by the integral curves of the vector field $X_\lambda$

$$(\lambda_2 z_1 (z_2 - z_1) + \lambda_3 z_1 z_2) \frac{\partial}{\partial z_1} + (\lambda_1 z_2 (z_1 - z_2) + \lambda_3 z_1 z_2) \frac{\partial}{\partial z_2}$$
Using E. Paul’s work we can assure that the fibers of

\[ F_\lambda(z_1, z_2) = z_1^{\lambda_1} z_2^{\lambda_2} (z_2 - z_1)^{\lambda_3} \]

over \( \mathbb{C}^* \) are the leaves of \( \mathcal{F}_\lambda \) in \( \mathbb{C}^2 \setminus \{z_1 z_2(z_2 - z_1) = 0\} \). In other words, \( F_\lambda \) is a \textit{first integral} for \( \mathcal{F}_\lambda \).
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over $\mathbb{C}^*$ are the leaves of $\mathcal{F}_\lambda$ in $\mathbb{C}^2 \setminus \{z_1 z_2 (z_2 - z_1) = 0\}$. In other words, $F_\lambda$ is a first integral for $\mathcal{F}_\lambda$. Hence, there are 3 types of leaves: those in $F_\lambda^{-1}(0)$ (the tangent cone) and $F_\lambda^{-1}(t)$ with $t \in \mathbb{C}^*$ (generic leaf, denoted $L$).
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In other words, the leaves of $\mathcal{F}_\lambda$ have a “natural” translation surface structure. Notation $(L, \eta|_L)$
Write $X_\lambda = X_1 + iX_2$ and define $\mathcal{F}_0$ to be the (real) foliation on $\mathbb{C}^2$ defined by the integral curves of $X_1$. 
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**Theorem.**

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Dictionary

Ferrán Valdez
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2. If we change $L$ for $\rho e^{i\alpha}L$, then $\mathcal{F}_0$ restricted to $\rho e^{i\alpha}L$ is conjugated to the foliation on $S_P$ formed by geodesics parallel to $\theta + \alpha$. 
Sketch of proof.

Aim: construct \( \phi : (L, \eta) \rightarrow S_P \).
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Locally, $X_\lambda$ restricted to $L$ is of the form

$$t^{1-\lambda_2}(t-1)^{1-\lambda_3} \frac{\partial}{\partial t}, \quad t \in \mathbb{H}$$
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Therefore

$$z(t) := \int^t \xi^{\lambda_2-1}(\xi - 1)^{\lambda_3-1} d\xi$$

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“Hard” part: show that \(z\) extends to define \(\phi\). \(\square\)
Remarks

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$$2X_1 = \left[ \lambda_2(x_1^2 - y_1^2) - (\lambda_2 + \lambda_3)(x_1x_2 - y_1y_2) \right] \frac{\partial}{\partial x_1} + \left[ 2\lambda_2 x_1 y_1 - (\lambda_2 + \lambda_3)(x_1y_2 + x_2y_1) \right] \frac{\partial}{\partial y_1} + \left[ \lambda_1(x_2^2 - y_2^2) - (\lambda_1 + \lambda_3)(x_1x_2 - y_1y_2) \right] \frac{\partial}{\partial x_2} + \left[ 2\lambda_1 x_2 y_2 - (\lambda_1 + \lambda_3)(x_1y_2 + x_2y_1) \right] \frac{\partial}{\partial y_2}$$

has a periodic orbit.
The phase space of the billiard is three dimensional!
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Let $\mathcal{G}_\lambda$ be the foliation defined in homogeneous coordinates by:

$$\alpha := i_R i_{x_1} i_{x_2} (dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2)$$

The singular locus of $\mathcal{G}_\lambda$ is $\pi(z_1 z_2(z_2 - z_1) = 0)$
Foliations in $\mathbb{RP}(3)$

The Borromean rings "=" $\text{Sing} \mathcal{G}_\lambda$. 
Let $L \in \mathcal{F}_\lambda$ be fixed. If there exist $n, m \in \mathbb{Z}$ such that $n\lambda_i + m\lambda_j - \frac{1}{2} \in \mathbb{Z}$, then

$$\pi| : L \to \pi(L)$$

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Local picture near $\text{Sing} G_\lambda$: (up to analytic change of coordinates)

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Global picture near $\text{Sing} G_\lambda$. Let

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\pi : \widetilde{\mathbb{RP}(3)} \rightarrow \mathbb{RP}(3)
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be the blow-up along $\text{Sing}(G_\lambda)$ and $\widetilde{G}_\lambda$ the foliation defined by $\pi^* \alpha$. Each component of the Borromean rings defines a "torus" $T_i$. Let $(x, y)$ be local coordinates for $T_i$, then the trace of a leaf $L \in \widetilde{G}_\lambda$ in $T_i$ is of the form:

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\lambda_i x + (1 - \lambda_i) y = k, \quad k \in \mathbb{R}
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In other words, $\widetilde{G}_\lambda|_{T_i}$ is defined by "parallel lines".
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Let $\mathcal{G}_0$ be the foliation on $\mathbb{RP}(3)$ defined by $X_1$. Remark that $\text{Sing}(\mathcal{G}_0) \subset \text{Sing}(\mathcal{G})$. Since $X_1$ restricted to the tangent cone is conjugated to $\text{Re}(z^2 \frac{\partial}{\partial z})$, $\text{Sing}(\mathcal{G}_0)$ is given by:

\[ p_1 = [1 : 0 : 0 : 0] \quad p_2 = [0 : 0 : 1 : 0] \quad p_3 = [1 : 0 : 1 : 0] \]
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Remark the invariant (local) manifold $\{x = 0\}$!
¡Muchas gracias a todos y feliz Navidad!

vielen Dank und schöne Weihnachten!