Examples for Veech groups of origamis

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1 Introduction

An origami is a combinatorial object (see section 2) that defines a translation surface (i.e. all transition maps are translations) of some genus g. Using this one can construct a certain affine complex curve in the moduli space M_g of regular projective complex curves of genus g. This curve is a *Teichmüller curve*, i.e. the image of a complex geodesic in the corresponding Teichmüller space T_g. In general such a Teichmüller curve is defined up to birationality by a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, the *Veech group*. In the case of origamis the Veech group is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

In this article we study such origami Veech groups. We calculate the Veech groups for some infinite sequences of origamis. In particular we show that the well known congruence groups

$$\pm\Gamma_1(2k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | a \equiv \pm 1, b \equiv 0, d \equiv \pm 1 \bmod 2k \right\}$$

occur for odd $k \in \mathbb{N}$. Furthermore, we construct for each $g \geq 2$ origamis of genus g with Veech group $\pm\Gamma(2)$. This shows in particular that in each moduli space M_g ($g \geq 2$) there is a Teichmüller curve defined by an origami that is birational to the projective line without three points. With a variant of this construction one obtains a projective line without two points in M_g.

We state the basic definitions that we use and our main tool in section 2. In section 3 we give some helpful properties for calculating Veech groups. Finally, in sections 4 and 5 we obtain the results listed above.

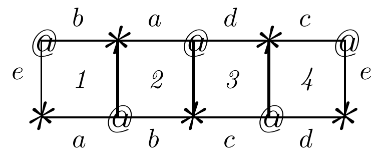
Acknowledgements: I would like to thank Frank Herrlich, my supervisor, for helpful discussions and suggestions, and Pierre Lochak for useful comments.

2 Veech groups of Origamis

We follow mostly the notations used in [S]. For more details and further information about the general context of origamis (also called *square tiled surfaces*) and Teichmüller curves see e.g. the references therein.

An *origami* can be defined as follows: Take finitely many copies of the euclidian unit square and glue them such that each left edge of a square is identified with a right edge and each upper edge with a lower one and vice versa. Thus one obtains a closed surface X. We restrict to the cases where this surface is connected. This definition by giving simple rules that define a combinatorial object gave rise to the name *origami* introduced in [L].

Example 1. *: An origami with 4 squares:*



After identifying edges labeled by same letter one obtains a closed surface X of genus 2: it is divided into 4 squares with 8 edges (after identification) and the two vertices $*$ and @. Hence the Euler characteristic is -2 and the genus is 2.

If one numbers the squares then the origami is given by two permutations σ_a and σ_b, where σ_a and σ_b indicate how the vertical and the horizontal edges are glued. In example 1 we have $\sigma_a = (1\ 2\ 3\ 4)$ and $\sigma_b = (1\ 2)(3\ 4)$.

The images of the squares on the surface X define a covering p from X to the torus E, where E is obtained as origami by glueing opposite sides of **one** square. The vertices of this one square define one marked point ∞ on E and the covering $p : X \to E$ is ramified at most over this point ∞. The degree of the covering is the number of squares and the preimages of ∞ on X are the vertices of the square tiling. In example 1 we have a covering of degree 4 ramified in the two points $*$ and @.
If $E^* := E - \{\infty\}$ and $X^* := X - p^{-1}(\infty)$, then $p : X^* \to E^*$ is a finite unramified covering of the punctured torus. Conversely, each finite unramified covering $p : X^* \to E^*$ defines an origami.

By the universal covering theorem, the fundamental group $\pi_1(X^*)$ of X^* is embedded via p into the fundamental group $\pi_1(E^*)$ of E^*. The group $\pi_1(E^*)$ is isomorphic to $F_2 := F_2(x, y)$, the free group in the two generators x and y. We take the homotopy class of a horizontal closed path on E^* to be x and that of a vertical one to be y.
The subgroup $U \cong \pi_1(X^*)$ of F_2 that we obtain in this way in example 1 – if we choose the base point in the first square – is $U =< xy, yx^{-1}, x^2yx^{-3}, x^3yx^{-2}, x^4 >$. The given figure is simply connected, thus paths corresponding to the identifications of edges define a set of generators for the fundamental group.

An origami defines *translation structures* - i.e. an atlas of charts such that all transition maps are translations - on the surface X^* as follows: Take a translation structure on E^* and lift it via p. For the purpose of this paper we may restrict to the translation structure on E^* which one obtains by identifying E^*

with $\mathbb{C}/\Lambda_0$ where Λ_0 is the lattice $\mathbb{Z} \oplus \mathbb{Z}i$ in $\mathbb{C}$.

For an origami O one calls

$$\mathrm{Aff}^+(O) := \{f : X^* \to X^* | f \text{ orientation preserving affine diffeomorphism}\}$$

the *affine group*, where X^* is obtained as above and the diffeomorphisms are affine with respect to the translation structure defined as above. Here *affine* means that the diffeomorphism is locally defined as a real-affine map of $\mathbb{C} \cong \mathbb{R} \oplus \mathbb{R}i$

$$x+iy \mapsto A*(x+iy)+t := (ax+by)+(cx+dy)i+t,\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), t \in \mathbb{C}$$

Since all transition maps in the atlas of X^* are translations, the matrix A is independent of the chart and we obtain a homomorphism

$$\mathbf{der} : \mathrm{Aff}^+(O) \to \mathrm{SL}_2(\mathbb{R}), f \mapsto A \quad (\text{ with } A \text{ as above}).$$

The image $\Gamma(O) := \mathbf{der}(\mathrm{Aff}^+(O))$ of the affine group is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ ([V]) called *Veech group*. It is the object we study in this article.

The Veech group depends on the choice of the lattice Λ_0 that we made above only up to conjugacy by a matrix in $\mathrm{SL}_2(\mathbb{R})$.
Changing the lattice does not only change the translation structure on the surface but also the complex structure on X defined by extending the translation atlas on X^* to a holomorphic atlas on X. Thus variation through all possible lattices defines a subset of the moduli space M_g which is in fact an affine curve, the *Teichmüller curve* mentioned in the introduction. This curve is birationally equivalent to the quotient of $\mathbb{H}$ by the action of the Veech group (acting as fuchsian group) (see e.g. [EG], [McM]).

The Veech group of an origami is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ ([GJ]). This can be seen e.g. from the characterization of Veech groups of origamis given in [S]. We will state it in the following theorem, since it is the main tool we use in this article. We use the natural projection

$$\begin{array}{rcl} \hat{\beta} : \mathrm{Aut}^+(F_2) & \to & \mathrm{Out}^+(F_2) \cong \mathrm{SL}_2(\mathbb{Z}) \\ \gamma & \mapsto & A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

where $a := \sharp_x(\gamma(x))$, i.e. the number of occurrences of x in $\gamma(x)$, where $\gamma(x)$ is viewed as word in x and y (with x^{-1} counted negative!) and similarly $b := \sharp_x(\gamma(y))$, $c := \sharp_y(\gamma(x))$ and $d := \sharp_y(\gamma(y))$. Furthermore, for a subgroup U of F_2 we denote by

$$\mathrm{Stab}(U) := \{\gamma \in \mathrm{Aut}^+(F_2) | \gamma(U) = U\}$$

the stabilizer of U in $\mathrm{Aut}^+(F_2)$.

Theorem 2. *Let O be an origami and the subgroup U of F_2 be defined by O as explained above. Then one has*

$$\Gamma(O) = \hat{\beta}(\text{Stab}(U))$$

3 A few properties of the stabilizer group

In this section we list some properties of the stabilizer group that we will use in the next sections. By $N \trianglelefteq H$ we denote that N is a normal subgroup of H.

Let U be a subgroup of F_2. Then U defines three subgroups of F_2 as follows:

$$\begin{aligned} \text{Norm}(U) &:= \{w \in F_2 | wUw^{-1} = U\}, \text{ the normalizer of } U \text{ in } F_2 \\ << U >>_{F_2} &:= < wuw^{-1} | \, w \in F_2, u \in U >, \text{ the normal closure of } U \text{ in } F_2 \\ \text{NT}(U) &:= \cap_{w \in F_2} wUw^{-1}, \text{ the biggest subgroup } N \text{ of } U \text{ that is normal in } F_2 \end{aligned}$$

The properties listed in the following remark are easily verified.

Remark 3. *Let U be a subgroup of F_2, U_i with $i \in I$ a family of subgroups of F_2 and γ an automorphism in $Aut^+(F_2)$. One has the following properties:*

1. *$Stab(U) \subseteq Stab(Norm(U))$,*
2. *$Stab(U) \subseteq Stab(<< U >>_{F_2})$*
3. *$\bigcap_{i \in I} Stab(U_i) \subseteq Stab(\bigcap_{i \in I} U_i)$*
4. *$Stab(\gamma(U)) = \gamma \circ Stab(U) \circ \gamma^{-1}$*
5. *$Stab(U) \subseteq Stab(NT(U))$*

Let now O be an origami, $p : X^* \to E^*$ the unramified covering and U the finite index subgroup of F_2 defined in section 2. The groups $\text{Norm}(U)$, $<< U >>_{F_2}$ and $\text{NT}(U)$ are also finite index subgroups of F_2 and define origamis O_1, O_2 and O_3.
Again let $p_1 : X_1^* \to E^*$, $p_2 : X_2^* \to E^*$ and $p_3 : X_3^* \to E^*$ be the unramified coverings defined by these three origamis.
Then p_1 is the unramified covering of E^* of minimal degree such that it is covered normally by X^*, i.e. there exists a normal unramified covering $q_1 : X^* \to X_1^*$ with $p_1 \circ q_1 = p$.
Similarly, p_2 is the unramified covering of E^* of maximal degree that is normal and covered by X^*.
Finally, p_3 is the minimal unramified covering of E^* that factors through p by a normal q_3, i.e. there is a normal covering q_3 of X^* such that $p \circ q_3 = p_3$.

The properties of stabilizer groups listed above imply the following corollary 4 to theorem 2.

Corollary 4. *With the definitions of the last paragraph we have: The Veech group $\Gamma(O)$ of the origami O is contained in the Veech groups $\Gamma(O_1)$, $\Gamma(O_2)$ and $\Gamma(O_3)$ of the origamis O_1, O_2 and O_3.*

4 X-origamis

In this section we study a sequence O_k of origamis we call *X-origamis* because of their shape (see figure 1). We detect their Veech groups $\Gamma(O_k)$ as congruence groups of level $2k$.

Definition 5. *Let O_k be the origami with $2k$ squares given in figure 1, i.e. the origami defined by the permutations*

$$\sigma_a := (1\ 2\ \ldots\ 2k) \in S_{2k} \text{ and } \sigma_b = ((1\ 2)(3\ 4)\ldots(2k-1\ \ 2k)) \in S_{2k}.$$

Recall from section 2 that σ_a gives the horizontal and σ_b the vertical identifications of the edges.

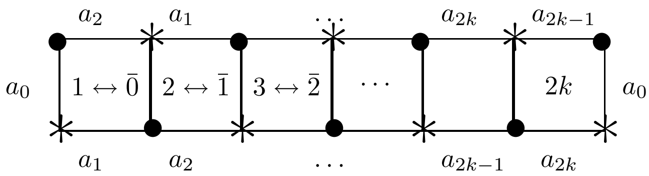


Figure 1

As in example 1 edges with same labels are identified. We obtain a closed surface X_k. It is divided into $2k$ squares with $4k$ edges and two vertices $\bullet$ and $*$. The genus of X_k is k. Recall from section 2 that O_k defines an unramified covering $p_k : X_k^* \to E^*$ of degree $2k$. The fundamental group $U_k = \pi_1(X_k^*) \subseteq \pi_1(E^*) = F_2$ is – if we choose the base point in the first square:

$$U_k = < x^{2k}, xy, yx^{-1}, x^2yx^{-3}, x^3yx^{-2}, \ldots, x^{2k-2}yx^{-(2k-1)}, x^{2k-1}yx^{-(2k-2)} >$$

Proposition 6. *The Veech group of O_k is*

$$\Gamma(O_k) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | 2b \equiv 0,\ a+b \equiv \pm 1 \text{ mod } 2k \text{ and } a+c \equiv b+d \equiv 1 \text{ mod } 2\}.$$

In particular we have

- *k odd $\Rightarrow \Gamma(O_k)$ is conjugated to $\pm\Gamma_1(2k)$ (defined as in the introduction).*
- *k even $\Rightarrow \Gamma(O_k)$ has the same index as $\pm\Gamma_1(2k)$ but is not conjugated.*

Proof. The proof is divided into the following steps:

1. One obtains for the baby origami O_1 with two squares:

$$\Gamma(O_1) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | a+c, b+d \text{ odd } \}$$

   and all Veech groups are contained in the first one, i.e. $\Gamma(O_k) \subseteq \Gamma(O_1)$.

2. The group U_k can be described alternatively as

$$U_k = \{ w \in F_2 | \sharp_x(w) + \Delta_y \text{ is divisible by } 2k \}.$$

   (precise definitions see below)

3. We solve the problem first in the principal congruence group $\Gamma(2)$:

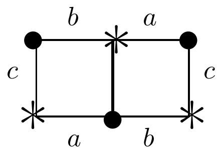
$$\Gamma(O_k) \cap \Gamma(2) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2) | 2b \equiv 0, \ a+b \equiv \pm 1 \mod 2k \}$$

4. Using 3. we show that $\Gamma(O_k)$ is the group claimed in the proposition.

5. Using 4. we show that $\Gamma(O_k)$ has the same index in $\mathrm{SL}_2(\mathbb{Z})$ as $\pm\Gamma_1(2k)$ and they are conjugated iff k is odd.

1.:

We consider the first element of the sequence O_1:



The corresponding subgroup of F_2 is

$$U_1 = < x^2, xy, yx^{-1} > = < x^2, xy, y^2 > = \{ w \in F_2 | \ \mathrm{le}(w) \text{ is even} \},$$

where $\mathrm{le}(w)$ denotes the length of w as word in x and y.
Hence for an automorphism $\gamma \in \mathrm{Aut}^+(F_2)$ we have:

$$\begin{aligned} \gamma(U_1) = U_1 \quad &\Leftrightarrow \quad \gamma \text{ preserves the parity of the length of words} \\ &\Leftrightarrow \quad \mathrm{le}(\gamma(x)) \text{ and } \mathrm{le}(\gamma(y)) \text{ are odd} \\ &\Leftrightarrow \quad \sharp_x(\gamma(x)) + \sharp_y(\gamma(x)) \text{ odd and } \sharp_x(\gamma(y)) + \sharp_y(\gamma(y)) \text{ odd} \\ &\Leftrightarrow \quad a+c \text{ and } b+d \text{ are odd for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \hat{\beta}(\gamma), \\ &\qquad\qquad \text{with } \hat{\beta} \text{ defined before theorem 2} \end{aligned}$$

By Theorem 2 the Veech group $\Gamma(O_1) = \hat{\beta}(\text{Stab}(U_1))$. This proves the first part of 1.

Furthermore, U_1 contains U_k for all k and it is the normal closure of U_k in F_2, i.e. $U_1 = << U_k >>_{F_2}$. This can be seen by checking the three generators of U_1: y^2 and xy are already elements of U_k and $x^2 = x(xy)x^{-1}(xy^{-1})$ with xy and xy^{-1} in U_k is also in $<< U_k >>_{F_2}$.
Using corollary 4 we obtain the second part of 1.

<u>2.:</u>

We identify the $2k$ squares of the origami O_k with the elements of

$$\mathbb{Z}/2k\mathbb{Z} = \{\bar{0}, \bar{1}, \ldots, \overline{2k-1}\}.$$

The group F_2 acts from the right on the set of the squares as follows: For $w \in F_2 = \pi_1(E^*)$ and $\bar{a}$ one of the squares, lift the path w on E^* via the covering p_k to a path on X_k^* with its starting point in the square $\bar{a}$. Let $\bar{b}$ be the square in which the ending point of the lifted path lies. Then define

$$\bar{a} \cdot w := \bar{b}.$$

Since we had chosen the base point for $U_k = \pi_1(X_k^*)$ in the square $\bar{0}$ one has for v, w in F_2 by definition:

$$w \in U_k \Leftrightarrow \bar{0} \cdot w = \bar{0} \qquad \text{and} \qquad vwv^{-1} \in U_k \Leftrightarrow \bar{b} \cdot w = \bar{b} \text{ with } \bar{b} := \bar{0} \cdot v \quad (1)$$

Let $\bar{a}$ be in $\mathbb{Z}/2k\mathbb{Z}$, then x, y, x^{-1}, y^{-1} act on $\bar{a}$ in the following way:

$$\begin{array}{rclrcl} \bar{a} \cdot x &=& \overline{a+1} & \bar{a} \cdot x^{-1} &=& \overline{a-1} \\ \bar{a} \cdot y &=& \begin{cases} \overline{a+1}\,, & \text{if } \bar{a} \text{ even} \\ \overline{a-1}\,, & \text{if } \bar{a} \text{ odd}\,. \end{cases} & \bar{a} \cdot y^{-1} &=& \begin{cases} \overline{a+1}\,, & \text{if } \bar{a} \text{ even} \\ \overline{a-1}\,, & \text{if } \bar{a} \text{ odd}\,. \end{cases} \end{array} \quad (2)$$

Here we use that x^{2k} is in U_k.

Now, we obtain the action of any w in F_2 on $\mathbb{Z}/k\mathbb{Z}$: Each $x^{\pm 1}$ contributes ± 1, each $y^{\pm 1}$ contributes 1 or -1 depending on the parity of the position of $y^{\pm 1}$ in w.

Definition 7. *For $w \in F_2$ let $\sharp_{|y|}(w|$ odd$)$ be the total number of occurrences of y and y^{-1} in w at an odd position (y^{-1} counted positive!). Similarly, denote by $\sharp_{|y|}(w|$ even$)$ the number of occurrences of y and y^{-1} in w at an even position. Furthermore define*

$$\Delta_y(w) := \sharp_{|y|}(w|\ \textit{odd}) - \sharp_{|y|}(w|\ \textit{even}).$$

E.g. for $w := xyxy^{-1}x^2y^{-1}$ one has $\sharp_{|y|}(w|\text{ odd}) = 1$, $\sharp_{|y|}(w|\text{ even}) = 2$ and $\Delta_y(w) = -1$.

Using (2) we obtain

$$\bar{a} \cdot w = \begin{cases} \overline{a + \sharp_x(w) + \Delta_y(w)}\ , \text{ if } \bar{a} \text{ even} \\ \overline{a + \sharp_x(w) - \Delta_y(w)}\ , \text{ if } \bar{a} \text{ odd} \ . \end{cases} \quad (3)$$

Since $U_k = \{w \in F_2 | \, \bar{0} \cdot w = \bar{0}\}$, 2. follows from (3).

<u>**3.:**</u>

Before restricting to $\Gamma(2)$ we stay in the general setting and observe that it is sufficient to consider the two generators xy and y^2. More precisely: For $\gamma \in \mathrm{Aut}^+(F_2)$

$$\gamma \in \mathrm{Stab}(U_k) \Leftrightarrow \gamma \in \mathrm{Stab}(U_1) \text{ and } \gamma(y^2), \gamma(xy) \in U_k \quad (4)$$

$\Rightarrow$ follows by 1. and the definition of $\mathrm{Stab}(U_k)$.
$\Leftarrow$ is true since U_k is the subgroup of U_1 consisting of those words in the three generators $w_1 := x^2$, $w_2 := xy$ and $w_3 := y^2$ of U_1 for which the number of occurrences of w_1 is divisible by k (w_1^{-1} counted negative), i.e.

$$U_k = \{w = w(w_1, w_2, w_3) \in U_1 | \, \sharp_{w_1}(w) \text{ is divisible by } k\}.$$

U_k is generated as normal subgroup of U_1 by w_1^k, w_2 and w_3. Thus it is sufficient to check x^{2k}, xy and y^2 in order to find out, if a given $\gamma \in \mathrm{Stab}(U_1)$ fixes U_k. But $\gamma(x^{2k}) = \gamma(w_1)^k$ and the number of occurrences of each generator in it is divisible by k. Hence it follows (4).

As next step observe that in order to check whether A is in $\Gamma(O_k)$ it is sufficient to consider one preimage of A under $\hat{\beta}$: Let A be in $\mathrm{SL}_2(\mathbb{Z})$ and $\gamma_0 \in \mathrm{Aut}^+(F_2)$ such that $\hat{\beta}(\gamma_0) = A$. Since $\hat{\beta}$ is the quotient map $\mathrm{Aut}^+(F_2) \to \mathrm{Out}^+(F_2) \cong \mathrm{SL}_2(\mathbb{Z})$, an automorphism γ is mapped to A by $\hat{\beta}$ iff it is conjugated to γ_0. Thus we have :

$$\begin{aligned} A \in \Gamma(O_k) &\overset{\text{thm.2}}{\Leftrightarrow} \exists \gamma \in \mathrm{Stab}(U_k) : \hat{\beta}(\gamma) = A \Leftrightarrow \exists w \in F_2 : w\gamma_0 w^{-1} \in \mathrm{Stab}(U_k) \\ &\overset{(1)}{\Leftrightarrow} \exists \bar{b} \in \mathbb{Z}/(2k\mathbb{Z}) : \bar{b} \cdot \gamma_0(u) = \bar{b} \text{ for all } u \in U_k. \end{aligned} \quad (5)$$

Observe that $\mathrm{Norm}(U_k) = U_1$ (by checking that it contains the three generators of U_1). Thus $\mathrm{Norm}(U_k)$ has index 2 in F_2 and $1, x$ are coset representatives. Thus in (5) it is sufficient to consider $\bar{b} \in \{\bar{0}, \bar{1}\}$. Together with (4) it follows that

$$\begin{aligned} A \in \Gamma(O_k) \Leftrightarrow \quad & (\bar{0} \cdot \gamma_0(y^2) = \bar{0} \text{ and } \bar{0} \cdot \gamma_0(xy) = \bar{0}\,) \text{ or} \\ & (\bar{1} \cdot \gamma_0(y^2) = \bar{1} \text{ and } \bar{1} \cdot \gamma_0(xy) = \bar{1}\,). \end{aligned} \quad (6)$$

Now, suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2) =< A_1 := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, A_2 := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, A_3 := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} > . \quad (7)$$

We define the three automorphisms

$$\gamma_1 : x \mapsto x, y \mapsto x^2y \qquad \gamma_2 : x \mapsto xy^2, y \mapsto y \qquad \gamma_{-I} : x \mapsto x^{-1}, y \mapsto y^{-1}$$

and set

$$G(2) :=< \gamma_1, \gamma_2, \gamma_{-I} > .$$

Thus $\hat{\beta}(G(2)) = \Gamma(2)$. Let γ_0 be in $G(2)$ with $\hat{\beta}(\gamma_0) = A$.
We will use in the following the fact proven below in lemma 8 that $G(2)$ respects Δ_y, i.e.:

$$\forall \gamma \in G(2), w \in F_2 : \Delta_y(\gamma(w)) = \Delta_y(w).$$

Then we have:

$$\bar{0} \cdot \gamma(xy) \stackrel{(3)}{=} \overline{0 + \sharp_x(\gamma(xy)) + \Delta_y(\gamma(xy))} \stackrel{\text{lem.8}}{=} \overline{\sharp_x(\gamma(x)) + \sharp_x(\gamma(y)) + \Delta_y(xy)} = \overline{a+b-1}.$$

Similarly one obtains

$$\begin{aligned} \bar{1} \cdot \gamma(xy) &= \bar{1} + \overline{a+b+1} \\ \bar{0} \cdot \gamma(y^2) &= \overline{2b} \quad \text{and} \quad \bar{1} \cdot \gamma(y^2) = \bar{1} + \overline{2b} \end{aligned}$$

Thus by (6) $A \in \Gamma(O_k)$ iff $2b \equiv 0$ and $a + b \equiv \pm 1$ modulo $2k$. This proves 3.

4.:

Recall that by 1. the Veech group $\Gamma(O_k)$ is a subgroup of $\Gamma(O_1)$. Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\Gamma(O_1)\backslash\Gamma(2)$. The index of $\Gamma(2)$ in $\Gamma(O_1)$ is 2, since by 1. any element of $\Gamma(O_1)$ maps to either $\bar{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\bar{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})/\Gamma(2) = \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$. Therefore A has a decomposition $A = B \cdot S$ for some matrix B in $\Gamma(2)$. We define the automorphism

$$\gamma_s : x \mapsto y, y \mapsto x^{-1},$$

then γ_s is a preimage of S under $\hat{\beta}$. Furthermore, we take a preimage γ_B of B in $G(2)$, then $\gamma_A := \gamma_B \circ \gamma_S$ is a preimage of A. One obtains:

$$\bar{0} \cdot \gamma_A(xy) \quad \stackrel{(3)}{=} \quad \overline{0 + \sharp_x(\gamma_A(xy)) + \Delta_y(\gamma_B(yx^{-1}))} \stackrel{Lem8}{=} \overline{a+b+1}$$

Similarly, one calculates $\bar{0}\cdot\gamma_A(y^2)$, $\bar{1}\cdot\gamma_A(xy)$ and $\bar{1}\cdot\gamma_A(y^2)$ and obtains altogether:

$$\begin{aligned} \bar{0} \cdot \gamma_A(xy) &= \overline{a+b+1} & \qquad \bar{0} \cdot \gamma_A(y^2) &= \overline{2b} \\ \bar{1} \cdot \gamma_A(xy) &= \bar{1} + \overline{a+b-1} & \qquad \bar{1} \cdot \gamma_A(y^2) &= \bar{1} + \overline{2b} \end{aligned}$$

Thus it follows that $A \in \Gamma(O_k)$ iff $2b \equiv 0 \bmod 2k$ and $a + b \equiv \mp 1 \bmod 2k$. Together with 1. and 3. this finishes the proof of 4.

5.:

In order to obtain that $\Gamma(O_k)$ and $\pm\Gamma_1(2k)$ have the same index in $\mathrm{SL}_2(\mathbb{Z})$, we use the fact that $\Gamma(2k)$ is contained in $\Gamma(O_k)$ as well as in $\pm\Gamma_1(2k)$. Therefore it is sufficient to show that their images in $\mathrm{SL}_2(\mathbb{Z}/2k\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z})/\Gamma(2k)$ have the same number of elements.
Using 4. we obtain that the image of $\Gamma(O_k)$ in $\mathrm{SL}_2(\mathbb{Z})$ is:

$$\{\begin{pmatrix} \pm1 & 0 \\ e & \pm1 \end{pmatrix}, \begin{pmatrix} \pm1+k & k \\ o & \pm1+k \end{pmatrix} | e, o \in \mathbb{Z}/2k\mathbb{Z},\ e \text{ even },o \text{ odd } \}, \quad \text{if } k \text{ odd} \tag{8}$$

$$\{\begin{pmatrix} \pm1 & 0 \\ e & \pm1 \end{pmatrix}, \begin{pmatrix} \pm1+k & k \\ e' & \pm1+k \end{pmatrix} | e, e' \in \mathbb{Z}/2k\mathbb{Z},\ e, e' \text{ even } \}, \quad \text{if } k \text{ even} \tag{9}$$

Thus the image has in both cases $4k$ elements. The image of $\pm\Gamma_1(2k)$ consists of $4k$ elements as well.

Observe by (9) that $\Gamma(O_k)$ is contained in $\pm\Gamma(2)$ if k is even. But $\Gamma(2)$ is normal and does not contain $\pm\Gamma_1(2k)$. Therefore $\Gamma(O_2)$ is not conjugated to $\pm\Gamma_1(2k)$ if k is even.
For k odd, one can check by a calculation in $\mathrm{SL}_2(\mathbb{Z}/2k\mathbb{Z})$ that

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \Gamma_1(2k) \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} = \Gamma(O_k). \tag{10}$$

□

Lemma 8. *The number $\Delta_y(w) = \sharp_{|y|}(w|\ odd) - \sharp_{|y|}(w|\ even)$ is invariant under $G(2) =< \gamma_1, \gamma_2, \gamma_{-I} >$, i.e. if γ is in $G(2)$, then*

$$\forall w \in F_2 : \Delta_y(\gamma(w)) = \Delta_y(w).$$

Proof. It is sufficient to check the claim for the generators of $G(2)$:

$$\gamma_1 : x \mapsto x, y \mapsto x^2y, \quad \gamma_2 : x \mapsto xy^2, y \mapsto y \quad \text{ and } \gamma_{-I} : x \mapsto x^{-1}, y \mapsto y^{-1}$$

Consider $\gamma := \gamma_1$: Let w be an arbitrary element in F_2, thus w is a reduced word in the four letters x,y, x^{-1}, y^{-1}: $w = w(x, y, x^{-1}, y^{-1})$ and $\gamma(w) = w(x, x^2y, x^{-1}, y^{-1}x^{-2})$.
Observe that for the words of replacement x, x^2y, x^{-1}, $y^{-1}x^{-2}$ the value of Δ_y is the same as for the original words x, y, x^{-1} and y^{-1}, their length is odd and that reduction also does not change the value of Δ_y. Hence $\Delta_y(\gamma_1(w)) = \Delta_y(w)$. With the same arguments this is true for γ_2 and γ_{-I}. Thus the claim holds. □

Using this sequence of origamis one can construct origamis having Veech group $\pm\Gamma_1(2k)$ (for k odd). In the following corollary, we use the automorphism $\gamma : x \mapsto x, y \mapsto x^{-k}y$.

Corollary 9. *Let k be odd. Define $V_k := \gamma(U_k)$ with the group U_k defined in proposition 6. Call P_k the origami that is defined by the finite index subgroup V_k of F_2. Then $\Gamma(P_k) = \pm\Gamma_1(2k)$.*

Proof. By remark 3 we have $\mathrm{Stab}(V_k) = \gamma \circ \mathrm{Stab}(U_k) \circ \gamma^{-1}$. By theorem 2 it follows that

$$\Gamma(P_k) = \hat{\beta}(\gamma)\Gamma(O_k)\hat{\beta}(\gamma^{-1}) = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \Gamma(O_k) \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \overset{(10)}{=} \pm\Gamma_1(2k)$$

□

5 Stair-origamis

In this section we consider two infinite sequences G_k and St_k of origamis of genus k. We show for both that all origamis in the sequence have the same Veech group. Because of their shape (see figures 2 and 3) they are called *stair origamis*.

The smallest example of the two sequences, the stairs with 3 and 4 squares, appear e.g. in [M], where the equations for the Teichmüller curves defined by these two origamis are calculated. The stair with three squares is because of its shape also called *L-origami* and is generalized in another sequence with origamis all in genus 2 (see e.g [HL], [S]).
The stairs with an odd number of squares occur in [H], where they are used to construct origamis that cover it having Veech group $\mathrm{SL}_2(\mathbb{Z})$.

Definition 10. *Let G_k be the origami with $2k$ squares ($k \geq 2$) in figure 2 given by the permutations*

$$\sigma_a := (1\ 2)\ \ldots\ (2k-1\ 2k) \text{ and } \sigma_b := (2\ 3)\ \ldots\ (2k-2\ 2k-1) \in S_{2k}$$

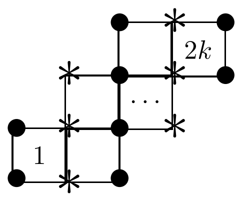


Figure 2

Here opposite edges are identified. One obtains a closed surface with the two marked points ● and ∗. Its genus is k. The fundamental group is

$$U_k = < \quad y, (xy)^{k-1}xyx^{-1}(xy)^{-(k-1)}, (xy)^j x^2 (xy)^{-j}, \\ (xy)^i xy^2 x^{-1}(xy)^{-i} | j \in \{0, \ldots, k-1\}, i \in \{0, \ldots, k-2\} \ >$$

Proposition 11. *The Veech group $\Gamma(G_k)$ is for all $k \in \mathbb{N}$ the principal congruence group $\Gamma(2)$.*

Proof. The proof is divided into two parts: In the first part we show $\Gamma(2)$ is a subgroup of $\Gamma(G_k)$; in the second part we show that it is not bigger.

$\Gamma(G_k)$ is a subgroup of $\Gamma(2)$:

Recall that the group $\Gamma(2)$ is generated by the three matrices A_1, A_2, A_3 given in (7). Take again the three preimages under $\hat{\beta}$:

$$\gamma_1 : \left\{ \begin{array}{ccc} x & \mapsto & x \\ y & \mapsto & x^2 y \end{array} \right. , \gamma_2 : \left\{ \begin{array}{ccc} x & \mapsto & xy^2 \\ y & \mapsto & y \end{array} \right. \text{ and } \gamma_3 : \left\{ \begin{array}{ccc} x & \mapsto & x^{-1} \\ y & \mapsto & y^{-1} \end{array} \right.$$

We show that $\gamma_i(U_k) = U_k$.

Observe that U_k contains $N :=<< x^2, y^2 >>_{F_2}$. More precisely, U_k is generated by N and the two elements y and cyc^{-1} with $c := (xy)^{k-1}x$.
Observe furthermore, that $\gamma_i(N) = N$ for $i = 1, 2, 3$:
E.g. $\gamma_1(x^2) = x^2 \in N$ and $\gamma_1(y^2) = x^2yx^2y = y((y^{-1}x^2y)x^2y^2)y^{-1} \in N$. This works similarly for $i = 2$ and $i = 3$. Thus we have $\gamma(N) = N$ for all $\gamma \in G(2)$.

Since $N \trianglelefteq F_2$ and $N \subseteq U_k$, it follows that

$$\forall n \in N, w, v \in F_2 : \ wnv = uwv \text{ with some } u \in U_k. \tag{11}$$

One obtains e.g.:
$\gamma_1(y) = x^2y \in U_k$ and
$\gamma_1(cyc^{-1}) = (\gamma_1(xy))^{k-1}xx^2yx^{-1}(\gamma_1(xy))^{-(k-1)} = (x^3y)^{k-1}xx^2yx^{-1}(x^3y)^{-(k-1)} \overset{(11)}{=} u(xy)^{k-1}xyx^{-1}(xy)^{-(k-1)} = ucyc^{-1}$ for some $u \in U$. Thus $\gamma_1(U_k) = U_k$.
This works similarly for $i = 2$, $i = 3$, which finishes the proof that $\Gamma(2) \subseteq \Gamma(G_k)$.

$\Gamma(2)$ is the whole group $\Gamma(G_k)$:

The matrices

$$\begin{aligned} I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ B_1 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ B_3 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \\ B_4 &= \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \ B_5 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

form a system of coset representatives of $\Gamma(2)$ in $\mathrm{SL}_2(\mathbb{Z})$. Thus it remains to show, that B_1, B_2, B_3, B_4 and B_5 are not in $\Gamma(G_k)$.
Observe that all generators and thus all elements of U_k contain an even number of occurrences of x. Since y is in U_k, the number $\sharp_x(\gamma(y))$ has to be even for an automorphism γ in $\mathrm{Stab}(U_k)$. This implies that the top right entry of an element of $\Gamma(G_k)$ has to be even. From this argument it follows that B_1, B_2, B_3 and B_4 are not in $\Gamma(G_k)$.
It remains to check B_5. We take the preimage $\gamma_0 : x \mapsto xy, y \mapsto y$ in $\mathrm{Aut}^+(F_2)$ of B_5 under $\hat{\beta}$.
Then we have for each other preimage $\gamma := w \cdot \gamma_0 \cdot w^{-1}$ ($w \in F_2$):

$\gamma(xy^{-1}xy^{-1}) = w\gamma_0(xy^{-1}xy^{-1})w^{-1} = wx^2w^{-1} \in N \subseteq U_k$.
But $xy^{-1}xy^{-1}$ is not in U_k, thus $\gamma \notin \mathrm{Stab}(U_k)$. From this it follows that $B_5 \notin \Gamma(G_k)$.

□

Definition 12. *Let St_k be the origami with $2k-1$ ($k \geq 2$) squares in figure 3 given by the permutations*

$$\sigma_a := (1\ 2)\ \ldots\ (2k-3\ 2k-2) \text{ and } \sigma_b := (2\ 3)\ \ldots\ (2k-2\ 2k-1) \in S_{2k-1}$$

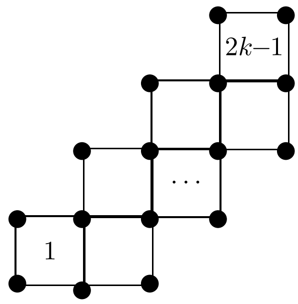


Figure 3

Again opposite edges are identified. One obtains a closed surface with one marked point: •. Its genus is k. The fundamental group is

$$U_k =< \quad y, (xy)^{k-1}x(xy)^{-(k-1)}, (xy)^j x^2 (xy)^{-j}, \\ (xy)^j xy^2x^{-1}(xy)^{-j} |\ j \in \{0, \ldots, k-2\} >$$

Proposition 13. *The Veech group $\Gamma(St_k)$ is for all $k \in \mathbb{N}$ the congruence group*

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | a+c \text{ and } b+d \text{ odd } \right\}.$$

Proof. We have

$$A \in \Gamma \Leftrightarrow A \text{ is sent to the image of } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

in $\mathrm{SL}_2(\mathbb{Z})/\Gamma(2) = \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ under the natural projection. Thus Γ is generated as normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ by $\Gamma(2)$ and the matrix B_2.
Take the automorphisms γ_1, γ_2 and γ_3 defined as in the proof of proposition 11 and take the automorphism $\gamma_4 : x \mapsto y, y \mapsto x^{-1}$ as preimage of B_2 under $\hat{\beta}$.
Observe that U_k again contains $N =<< x^2, y^2 >>_{F_2}$ and is generated by N and the two elements y and cxc^{-1} with $c := (xy)^{k-1}$.
We have already seen in the last proof that $\gamma_i(N) = N$ for $i \in \{1, 2, 3\}$ and it is easily seen that $\gamma_4(N) = N$. Furthermore, one can check similarly as in the last proof that $\gamma_i(y)$ and $\gamma_i(cyc^{-1})$ is in U_k. Hence Γ is contained in the Veech group of St_k.

Finally we show that $B_1 \notin \Gamma(St_k)$: Take one fixed preimage of B_1 under $\hat{\beta}$: $\gamma_5 : x \mapsto x, y \mapsto xy$. Then for each conjugated automorphism $\gamma := w\gamma_5 w^{-1}$ ($w \in F_2$) one has $\gamma(x^{-1}yx^{-1}y) = wy^2w^{-1} \in St_k$, but $x^{-1}yx^{-1}y \notin St_k$. Thus $\Gamma(St_k) \neq \mathrm{SL}_2(\mathbb{Z})$. It contains Γ which has index 3. Thus it is equal to Γ. □

Since $\mathbb{H}/\Gamma(2) \cong \mathbb{P}^1\backslash\{0, 1, \infty\}$ and $\mathbb{H}/\Gamma \cong \mathbb{P}^1\backslash\{0, 1\}$ – where Γ is from proposition 13 and the two groups act as fuchsian groups on $\mathbb{H}$ – we obtain the following result:

Corollary 14. *For each $g \geq 2$, there is an origami O of genus g such that $\mathbb{H}/\Gamma(O)$ is the affine line without three points and for each $g \geq 2$ there is an origami O such that $\mathbb{H}/\Gamma(O)$ is the affine line without 2 marked points.*

As mentioned in section 2 this implies in particular that the Teichmüller curve defined by these origamis is birationally to the projective line without three points for the stair-origamis G_k, respectively to the projective line without two points for St_k.

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