HOMEWORK 10

(1) In this exercise, we prove the following Theorem:

Theorem 0.1. *(Cheeger-Gromoll) Suppose (M, g) is a compact Riemannian manifold with* $\mathrm{Ric} \geq 0$*. Then the universal cover splits isometrically as a product $N \times \mathbb{R}^p$, where N is a compact manifold.*

Let $\tilde{M} = N \times \mathbb{R}^p$, where $\mathbb{R}^p$ is a maximal Euclidean factor (i.e., such that $\tilde{M}$ cannot also split as $N' \times \mathbb{R}^q$ for any $q > p$). We will show that N must be compact.

(a) Show that a geodesic $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \in N \times \mathbb{R}^p$ is a line iff either $\gamma_1(t), \gamma_2(t)$ are also lines (up to a constant multiple), or one is a line and the other is constant.

(b) Show that N cannot contain lines, and that every line has the form $\gamma(t) = (x, tv + w)$ for $v, w \in \mathbb{R}^p$ and $x \in N$.

(c) Prove that any $g \in \pi_1(M)$, acting on $\tilde{M}$ as a deck transformation, has the form of $g = (\phi_1, \phi_2)$, where $\phi_1 : N \to N$ is an isometry and $\phi_2 : N \times \mathbb{R}^p \to \mathbb{R}^p$.

Hint. *Isometries preserve lines, so $g(\{x\} \times \mathbb{R}^p) = \{x'\} \times \mathbb{R}^p$ for some x'...*

(d) **Suppose that N is not compact**. Let γ be a ray in N, and let $c(t) = \pi(\gamma(t), 0)$, where $\pi : \tilde{M} = N \times \mathbb{R}^p \to M$ is the covering map. Show that one can choose a sequence $\{t_i\}$, $t_i \to \infty$, such that $c'(t_i)$ converges to some vector $v \in TM$.

(e) Show that there is a sequence of deck transformations $\{g_i\}$ such that the geodesics $\gamma_i(t) = g_i(\gamma(t + t_i), 0)$ converge to a geodesic $\tilde{\gamma}(t)$ in $\tilde{M}$.

Hint. *Show that the sequence of vectors $\{\gamma_i'(0)\}$ converges.*

(f) Show that the limit geodesic is a line and it is tangent to N. Therefore we contradict point (b).

(2) Suppose (M, g) is a complete, compact Riemannian manifold with $\mathrm{Ric} \geq 0$. If M is $K(\pi, 1)$, i. e. the universal cover is contractible, then the universal covering is Euclidean space and (M, g) is a flat manifold.

(3) If (M, g) is compact with $\mathrm{Ric} \geq 0$ and has $\mathrm{Ric} > 0$ on some tangent space T_pM, then $\pi_1(M)$ is finite.