HOMEWORK 11

(1) As stated in the lecture, we can prove the Sphere Theorem using the Diameter Sphere Theorem (Grove-Shiohama). All we need is the following lemma of Klingenberg:

Lemma. *(Klingenberg 1961) Let M be a compact simply conneced Riemannian manifold with strictly $1/4$-pinched sectional curvature (i.e. $1 \leq \sec \leq 4$ up to rescaling).*

Then the injectivity radius $\operatorname{inj} M^n$ *and the conjugate radius* $\operatorname{conj} M^n$ *coincide:*

$$\operatorname{inj} M^n = \operatorname{conj} M^n \geq \pi/\sqrt{\max \sec_M}.$$

However, the proof of this lemma in general needs some prerequisites beyond our lecture. Therefore we prove it in a special case that $n = \dim M$ is even.

Lemma. *(Klingenberg 1956) Let M be an even dimensional compact simply connected Riemannian manifold with $\sec_M > \delta > 0$. Then the injectivity radius* $\operatorname{inj} M^n$ *and the conjugate radius* $\operatorname{conj} M^n$ *coincide:*

$$\operatorname{inj} M^n = \operatorname{conj} M^n \geq \pi/\sqrt{\max \sec_M}.$$

Fact. *Let M be a complete compact manifold. Then* $\operatorname{inj} M^n = \min\{\operatorname{conj} M^n, \frac{1}{2}\ell(\gamma)\}$*, where $\ell(M) = \inf_p \ell_M(p)$ is the minimal length of a nontrivial closed geodesic $c_0 : \mathbb{R}/\mathbb{Z} \to M$.*

(a) Let c_0 be a closed geodesic, and $P : (c_0'(0))^\perp \to (c_0'(0))^\perp$ be the parallel translation along c_0. Show that P has a fixed point.
(b) Suppose by contradiction that $\operatorname{inj} M < \operatorname{conj} M$. Using the fact above, prove that there exist a closed geodesic c_0 and a variation of curves c_t such that $2\ell(c_t) < \operatorname{inj} M$, for t nonzero and sufficiently small.
(c) Show for all $t > 0$, c_t lifts to a closed curve $\tilde{c}_t$ in $T_{c_t(0)}M \subseteq TM$, with $\tilde{c}_t(0) = 0$.
(d) Prove that c_0 can be lifted to a closed curve $\tilde{c}_0$ in $T_{c_0(0)}M$ with $\tilde{c}_0(0) = 0$, and get a contradiction.
(e) Prove the Sphere Theorem in even dimension using the Diameter Sphere Theorem.

(2) Let M be a complete, connected n-dimensional Riemannian manifold whose curvature and volume satisfy $\sec \geq \delta > 0$, and $\operatorname{Vol}(M) > \operatorname{Vol}(S^n(\delta))$. Prove that M is homeomorphic to a sphere.