HOMEWORK 9

(1) Let M be an open Riemannian manifold with $sec \geq 0$. Prove that any two souls of M are isometric and isotopic, and in particular homotopy equivalent.

(2) Let (M, g_i), $i = 1, 2$, be a Riemannian manifold endowed with complete metrics with sec ≥ 0, and let S_i, $i = 1, 2$, be a soul of M corresponding to g_i. Show that S_1, S_2 are homotopy equivalent, and $\dim S_1 = \dim S_2$.

(3) (a) Show that a distance nonincreasing function is volume increasing.
(b) Show that souls have minimum volume in their homotopy class.

Hint. *Take S_1 homotopy equivalent to a soul S, then $P(S_1) = S$, and $\operatorname{Vol} P(S_1) = \operatorname{Vol}(S)$.*

(4) In this exercise we aim to prove the following Lemma.

Lemma. *Let X be a compact metric space. Then any surjective distance non increasing map $f : X \to X$ is an isometry.*

(a) Show that f is continuous.
(b) Show that for all $x, y \in X$, $\epsilon > 0$, there exists an integer $N = N(x, y, \epsilon)$ such that for any $k > 0$, $d(f^N(x), f^N(y)) - d(f^{N+k}(x), f^{N+k}(y)) < \epsilon$.

Hint. *Given $x, y \in X$, check the sequence $(f^n(x), f^n(y))$ in $X \times X$.*

(c) Show that we can choose N depending only on ϵ which satisfies the condition of part (b).

Hint. *Cover $X \times X$ with balls and use continiuity.*

(d) Use surjectivity to prove the Lemma by contradiction.

Hint. *Take $a, b \in X$ with $d(f(a), f(b)) < d(a, b) - \delta$ and choose $x, y \in X$ such that $f^N(x) = a, f^N(y) = b$.*

(5) This exercise is about the notion of *pseudo soul* and some of its important properties.

Definition (Pseudo soul)**.** A submanifold of a open Riemannian manifold with nonnegative sectional curvature is called pseudo soul if it is isometric and isotopic to some soul.

Remember that given a soul S_0, the Sharafutdinov projection $P : M \to S_0$ is a distance non increasing projection (i.e. $P|_{S_0} = id$) and that there is a 1-parameter group of distance non increasing maps $P_t : M \to M$, $t \in [0, 1]$, such that $P_0 = id$ and $P_1 = P$.

(a) Use Exercise 1 to show that if a pseudo soul is isometric and isotopic to some soul S, then it is isotopic and isometric to any other soul.
(b) For any pseudo soul S, $P|_S$ is an isometry.
(c) For all pseudo soul S, $P_t|_S$ for all t is an isometry.
(d) For all pseudo soul S, $P_t(S)$ is totally geodesic. With an example show that it is not necessarily totally convex.

(6) Prove that pseudo souls are disjoint.