HOMEWORK 1

(1) Let M_κ be a Riemannian manifold with constant sectional curvature κ, and let $\gamma : \mathbb{R} \to M$ be a geodesic on M.
  (a) Prove that the curvature operator $R_v(x) = R(x,v)v$ satisfies $R_v(x) = \kappa(\|v\|^2 x - \langle x, v\rangle v)$.
  (b) Let $E(t)$ be a parallel vector field along γ, perpendicular to $\gamma'(t)$, and let f be a function. Prove that $J(t) = f(t)E(t)$ is a Jacobi field if and only if f solves the equation $f^{''} + \kappa f = 0$.
  (c) Compute the Jacobi fields along γ, in the cases $\kappa = -1, 0, 1$.

(2) Let $\gamma : \mathbb{R} \to M$ be a geodesic.
  (a) Prove that for any Jacobi field J along γ, the function $\frac{d}{dt}\langle J(t), \gamma^{'}(t)\rangle$ is constant.
  (b) Prove that if a Jacobi field satisfies $J(0) \perp \gamma^{'}(0), J^{'}(0) \perp \gamma^{'}(0)$, then $J(t) \perp \gamma^{'}(t)$ for all t.
  (c) Prove that the space $\mathcal{J}$ of Jacobi fields along γ splits as

$$\operatorname{span}\{\gamma^{'}, t\gamma^{'}\} \oplus \{J \mid J(t) \perp \gamma'(t) \quad \forall t\}.$$

(3) The goal of this excercise is to prove that distance function are uniquely characterized by the property of having a unit norm gradient.
  (a) Let S be a submanifold of M and $f(x) = \operatorname{dist}(S, x)$ be the distance function from S. Prove that whenever f is smooth, it satisfies $\| \nabla f \| = 1$. (*Hint:* use the first variation formula for the length function)
  (b) Let $f : M \to \mathbb{R}$ be a smooth function. Prove that if $\|\nabla f\|^2 = 1$ then the integral curves of f are geodesics.
  (c) Show that if a function $f : M \to \mathbb{R}$ satisfies $\|\nabla f\|^2 = 1$ then for any regular value z with level set $S = f^{-1}(z)$, the following is satisfied:

$$\operatorname{dist}(S, x) = |f(x) - z| \qquad \forall x \in S_2.$$

  In particular, the function $f - z$ is a (signed) distance function.

  By abuse of language, we can call distance function any function f such that $\|\nabla f\|^2 = 1$.

(4) Given $p \in M$ and $v \in T_pM$, let $\gamma_v : \mathbb{R} \to M$ be a geodesic with $\gamma_v(0) = p$, $\gamma_v'(0) = v$. Moreover, let J be a Jacobi vector field along γ such that $J(0) = 0, J^{'}(0) = w \in T_pM$. Show that

$J(t) = d_{tv}(exp_p)(tw)$.

(5) Show that $exp_p : T_pM \to M$ has a critical point at $v \in$ if and only if there exists a Jacobi field along γ_v such that $J(0) = J(\|v\|) = 0$.

(6) Prove that for any two Jacobi fields J_1, J_2 along a geodesic γ, the function $\langle J_1, {J_2}' \rangle$-$\langle J_1', J_2 \rangle$ is actually constant along γ.