HOMEWORK 4

(1) Let $S = (-1,1)^{n-1}$ with coordinates $s_1, \dots, s_{n-1}$. Let

$$\gamma : S \times \mathbb{R} \to M$$

be an $(n-1)$-parameter variation of geodesics with $S_t = \gamma(S \times \{t\})$ as equidistant hypersurfaces. For all $s \in S$, choose a parallel orthonormal frame $e_1(t), \dots, e_{n-1}(t)$ along $\gamma_s(t)$, and define $X_i(s,t) = d_{(s,t)}\gamma(\frac{\partial}{\partial s_i})$.

(a) Prove that $J_1(t) = X_1|_{\gamma_s(t)}, \dots, J_{n-1}(t) = X_{n-1}|_{\gamma_s(t)}$ are Jacobi fields satisfying $J_i'(t) = A_i J_i(t)$.

(b) For all $s \in S$, let $j_s(t) = \det\langle J_i(t), e_j(t)\rangle$. Prove that

$$Vol(S_t) = \int_S j_s(t)\, ds_1 \dots ds_{n-1}.$$

(2) In the following two problems we will see some applications of the Bishop-Gromov volume comparison theorem. Here, we aim to find a lower bound of the volume growth.

Theorem (Calabi–Yau). *Let (M^n, g) be a complete non-compact Riemannian manifold with $Ric \geq 0$. Then there exists a positive constant c depending only on p and n so that*

$$Vol(B_r(p)) \geq cr$$

for any $r \geq 2$.

We prove the Theorem in steps:

(a) Show that for any $p \in M$ there exists a geodesic $\gamma : [0,\infty) \to M$ with $\gamma(0) = p$ such that $dist(p, \gamma(t)) = t$ for $t > 0$.

(b) For any $t > \frac{3}{2}$, use the Bishop-Gromov volume comparison theorem to show that

$$\frac{Vol(B_{t+1}(\gamma(t)))}{Vol(B_{t-1}(\gamma(t)))} \leq \frac{(t+1)^n}{(t-1)^n}.$$

(c) Show that $B_1(p) \subset B_{t+1}(\gamma(t)) \setminus B_{t-1}(\gamma(t))$ and it follows

$$Vol(B_{t-1}(\gamma(t))) \leq C(n) Vol(B_1(p)) t,$$

where $C(n)$ is the infimum of the function $\frac{1}{t}\frac{(t-1)^n}{(t+1)^n-(t-1)^n}$ on $[\frac{3}{2}, \infty)$.

(d) Use the fact that $B_r(p) \supset B_{\frac{r+1}{2}-1}(\gamma(\frac{r+1}{2}))$ to get the theorem.

(3) The rigid version of Bishop-Gromov Theorem reads as follows:

Theorem. *Let M be an n-dimensional manifold with Ric $\geq (n-1)\,\kappa$ and let $\overline{M}_\kappa$ be the simply connected, n-dimensional manifold with* $\sec \equiv \kappa$. *If for some point $p \in M$, $\overline{p} \in \overline{M}_\kappa$ and $r > 0$ we have $Vol(B_r(p)) = Vol(\overline{B}_r(\overline{p}))$ then $B_r(p)$ and $\overline{B}_r(\overline{p})$ are isometric.*

Using Bishop-Gromov Theorem and its rigid version above, we now prove the following theorem, which describes what happens when the result in Bonnet-Myers Theorem is sharp.

Theorem. *Let (M, g) be a complete Riemannian manifold with Ric $\geq (n-1)k$ for some $k > 0$, and $diam(M, g) = \frac{\pi}{\sqrt{k}}$, then M is isometric to the standard sphere of radius $\frac{1}{\sqrt{k}}$.*

(a) For simplicity assume $k = 1$ and use Bishop-Gromov volume comparison theorem to show that for any point $p \in M$,
$$\frac{Vol(B_{\pi/2}(p))}{Vol(M)} \leq \frac{1}{2}.$$

(b) For $p, q \in M$ such that $d(p, q) = \pi$, show that $B_{\pi/2}(p)$ and $B_{\pi/2}(q)$ are both isometric to half sphere and then get the result.