HOMEWORK 5

(1) Let M be a Riemannian manifold with $sec \geq 1$, and $\frac{\pi}{2} < \operatorname{diam} M < \pi$. Let p, q be points in M such that $d(p,q) > \frac{\pi}{2}$.
  (a) Prove that $M \setminus B_{\frac{\pi}{2}}(p)$ is convex.
  (b) Prove that there exists a unique point $\bar{p}$ at maximal distance from p.

  Remark. These are the first, most important steps toward the proof of Berger's Diameter Sphere Theorem:

  Theorem. *(Diameter sphere theorem) Any manifold with $sec \geq 1$ and $diam > \frac{\pi}{2}$ is homeomorphic to a sphere.*

(2) **Busemann Functions.** Let M be a complete noncompact Riemannian manifold. Let $\gamma : [0, \infty) \to M$ be a unit speed ray, and define

$$b_\gamma(x) = \lim_{t \to \infty} (d(x, \gamma(t)) - t).$$

b_γ is called the Busemann function.
  (a) Show that for any $x \in M$ the function $d(x, \gamma(t)) - t$ is monotone decreasing, and therefore the limit $b_\gamma(x)$ is well defined.
  (b) Use Toponogov Comparison Theorem to show that if $sec \geq 0$, then the Busemann function is concave, i.e. for any geodesic segment $\alpha : [0,1] \to M$ between two points p_0, p_1, one has

$$b_\gamma(\alpha(t)) \geq (1-t) b_\gamma(p_0) + t b_\gamma(p_1)$$

  (c) For $p \in M$, let $R_p = \{\gamma \mid \gamma \text{ is ray, with } \gamma(0) = p\}$, and for every $\gamma \in R_p$, let $C_\gamma = b_\gamma^{-1}(0, \infty)$, and $C = \cap_{\gamma \in R_p} C_\gamma$. Use the previous point to show that C in convex.
  (d) Prove that C is compact.

  Hint. *By contradiction, then there exists $p_i \in C$ s.t. $d(p_i, p) \to \infty$, and find a ray γ such that $b_\gamma(p_i) < 0$*

(3) Prove that if a manifold satisfies $sec \geq k$ in the toponogov sense, then it satisfies $sec \geq k$ in the usual sense.