Abstract. We relate collapsing of Riemannian orbifolds and the rigidity theory of Seifert fiberings. Our results yield in particular a purely geometrical smooth characterization of infrasolv spaces which generalizes the almost flat manifold theorem of Gromov and Ruh from infranil to infrasolv manifolds and orbifolds. We show that, up to diffeomorphism, these spaces are precisely the ones which admit a collapse with bounded curvature and diameter to compact flat orbifolds. Moreover, we distinguish irreducible smooth fake tori geometrically from standard ones by proving that the former have non-vanishing $D$-minimal volume.

1. Introduction

In this article we consider smooth orbifolds $O = X/\Gamma$, where $X$ is a simply connected Riemannian manifold and $\Gamma$ is a properly discontinuously acting group of isometries. We also assume that a finite orbifold covering space of $O$ is a manifold. If $X$ is contractible, then $O$ is called an aspherical orbifold. For the orbifolds considered here, the notion of smooth map and diffeomorphism can be declared by considering equivariant smooth maps on $X$. In what follows we are interested in geometrical properties of orbifolds which distinguish their diffeomorphism class.

Our first result relates collapsing of Riemannian orbifolds and the theory of Seifert fiberings:

**Theorem 1.1.** Let $M$ be a closed Riemannian orbifold which admits a bounded curvature collapse to a compact aspherical Riemannian orbifold $O$. Then $M$ admits the structure of a smooth Seifert fibering over $O$ with infranil fiber. In particular, $M$ is an aspherical Riemannian orbifold.

Seifert fiber spaces with infranil fiber are a special class of orbibundles whose fibers are naturally diffeomorphic to infranil-manifolds. The
notion of Seifert fiber space with infranil fiber was systematically developed by Raymond, Lee and others. (See [17, 18] for recent accounts. The precise definition of Seifert fiber spaces is also given in Section 2.3 below.)

A Riemannian orbifold $S = X/\Gamma$ is called an infrasolv orbifold if $X$ is isometric to a simply connected solvable Lie group $S$ with left-invariant metric and $\Gamma$ is contained in the group of affine isometries of $S$. It is called infranil if $S$ is nilpotent. Remarkably, as is proved in [1], infrasolv orbifolds are diffeomorphic if and only if their fundamental groups are isomorphic.

**Theorem 1.2.** Let $S$ be a closed infrasolv manifold (or orbifold) with fundamental group $\pi_1(S)$. Then $S$ admits a canonical Seifert fibering with infranil fiber over a compact flat orbifold $O$.

Contracting the fibers of the above bundle to a point in an inhomogeneous way (see, for example, [22]) shows that $S$ admits a bounded curvature collapse to the flat orbifold $O$. Moreover, in this setting $S$ is infranil if and only if it collapses to a point.

We also prove the following converse to Theorem 1.2:

**Theorem 1.3.** Assume that the closed manifold (or compact orbifold) $M$ allows a bounded curvature collapse to a compact infrasolv orbifold $O$. Then $M$ is diffeomorphic to an infrasolv manifold (respectively, orbifold) $S$.

We thus obtain a purely geometrical smooth characterization of infrasolv manifolds as follows:

**Corollary 1.4.** A closed manifold is diffeomorphic to an infrasolv manifold if and only if it admits a bounded curvature collapse to a compact flat orbifold.

The corollary generalizes the famous almost flat manifold theorem of Gromov-Ruh [14, 21] from infranil to infrasolv manifolds. It also holds for orbi-instead of manifolds (cf. Section 4), and strengthens the topological characterizations of infranil and infrasolv manifolds in terms of collapsing, which were developed in [6] and in [22], to the smooth case.

The proof of the preceding results relies on techniques in the theory of collapsing of Riemannian manifolds (see the appendix of this article) and, as another crucial ingredient, on the rigidity of smooth Seifert fiberings with infranil fiber as established in [16]. We recall the
required rigidity results in Section 2.4.

The following applications shed some light on the geometry of fake tori. Recall that a smooth fake torus $T^n$ is a smooth manifold which is homeomorphic but not diffeomorphic to a standard torus $T^n$. The existence of smooth fake tori, $n \geq 5$, was established by Wall and Browder, see [3], [23, 15 A]. Since, as follows from [1], smooth fake tori do not carry an infrasolv structure, we first have:

**Corollary 1.5.** A smooth fake torus does not allow a bounded curvature collapse to a compact flat orbifold.

Gromov has defined the *minimal volume* of a smooth manifold $M$, $\text{MinVol}(M)$, as the infimum of all volumes $\text{vol}_g(M)$, where $g$ ranges over all smooth complete Riemannian metrics on $M$ whose sectional curvature is bounded in absolute value by one. Gromov’s critical volume conjecture asserts that there exists $\delta(n) > 0$ such that if a closed smooth $n$-manifold $M^n$ admits a metric with $|\sec(M, g)| \leq 1$ and $\text{vol}(M, g) \leq \delta(n)$, then $\text{MinVol}(M) = 0$. For $n \leq 4$ this conjecture is known to hold, see [20, 5] and the further references cited there.

Given a real number $D > 0$, one may also consider the *$D$-minimal volume*, $\text{D-MinVol}(M)$, where one requires that the infimum is taken over all metrics $g$ as above, which additionally have diameter bounded from above by $D$. Cheeger and Rong have shown that there exists $\delta = \delta(n, D) > 0$ such that if a closed smooth $n$-manifold $M$ admits a Riemannian metric $g$ with $|\sec(M, g)| \leq 1$, $\text{diam}(M, g) \leq D$, and $\text{vol}(M, g) \leq \delta(n, D)$, then $\text{D-MinVol}(M) = 0$, see [5]. In the special context of aspherical $M$ this was proved by Fukaya in [7].

Call a fake smooth torus *irreducible* if it is not a product of a standard torus and a fake torus of lower dimension. For example, smooth fake tori obtained by taking the connected sum of standard tori with exotic spheres are always irreducible. Then the following holds:

**Corollary 1.6.** Let $T$ be a smooth fake torus which is irreducible. Then, for all $D > 0$, there exists a positive constant $\nu(D)$ such that

$$\text{D-MinVol}(T) \geq \nu(D) > 0.$$ 

Notice that closed infrasolv manifolds have vanishing minimal volume, since, by Corollary 1.4, they actually have vanishing $D$-minimal volume for appropriate $D > 0$.

**Question 1.7.** Do smooth fake irreducible tori also always have non-vanishing minimal volume?
2. Seifert fiber spaces with nil-geometry

2.1. **Infra-\( G \) manifolds.** Let \( G \) be a Lie group. We let \( \text{Aff}(G) = G \cdot \text{Aut}(G) \) denote the group of affine transformations of \( G \). **Note that the affine group \( \text{Aff}(G) \) is precisely the normalizer of (the left-action of) \( G \) in the group of all diffeomorphisms \( \text{Diff}(G) \).**

**Definition 2.1.** Let \( \Delta \leq \text{Aff}(G) \) be a discrete subgroup whose homomorphic image in \( \text{Aut}(G) \) has compact closure. The quotient \( G/\Delta \) is called an **infra-\( G \) space**. If \( G/\Delta \) is a manifold then it is called an **infra-\( G \) manifold**.

**Example (Infranil manifolds).** Let \( N \) be a simply connected nilpotent Lie group. Then a compact infra-\( N \) manifold \( N/\Delta \) is called an **infranil manifold**. For an infranil manifold \( N/\Delta \), the intersection \( \Delta_0 = N \cap \Delta \) is the maximal nilpotent normal subgroup of \( \Delta \) and it has finite index in \( \Delta \). (Bieberbach’s first theorem, see [19, Chapter VIII].) In particular, for every infranil manifold, \( \Delta \) has finite image in \( \text{Aut}(N) \), and \( N/\Delta \) has a canonical finite normal covering by the nilmanifold \( N/\Delta_0 \).

2.1.1. **Affine group.** Let \( \nabla^\text{can} \) denote the natural flat connection on \( G \) which is defined by the property that left-invariant vector fields are parallel. The group of connection preserving diffeomorphisms \( \text{Aff}(G, \nabla^\text{can}) \) then coincides with the group of affine transformations \( \text{Aff}(G) \) of \( G \).

In particular, every infra-\( G \) manifold \( G/\Delta \) carries a flat connection \( \nabla^\text{can} \) induced from \( G \). The group of connection preserving diffeomorphisms \( \text{Aff}(G, \nabla^\text{can}) \) then coincides with the group of affine transformations \( \text{Aff}(G) \) of \( G \).

In particular, every infra-\( G \) manifold \( G/\Delta \)

2.2. **Infra-\( N \) bundles.** Let \( f : M \to B \) be a (locally trivial) fibration of smooth manifolds with fiber \( F \).

**Definition 2.2.** The fibration \( f \) is called an **infra-\( G \) bundle** over \( B \) if \( F = G/\Delta \) is an infra-\( G \) manifold and the structure group of \( f \) is contained in the affine group \( \text{Aff}(G/\Delta) \).

2.2.1. **Fiberwise affine maps.** Let \( f : M \to B \) and \( f' : M' \to B' \) be infra-\( G \) bundles with fiber \( G/\Delta \). Then a map \( \varphi : M \to M' \)
is called fiberwise affine if it is a morphism of $\text{Aff}(G/\Delta)$-bundles (that is, equivalently $\varphi$ is a bundle map which induces an affine diffeomorphism on each fiber). We let $\text{Aff}(M,G/\Delta) = \text{Aff}(M,G/\Delta,f)$ denote the group of all fiberwise affine diffeomorphisms of $f$.

2.3. Seifert fiber spaces. The following notion of Seifert fiber space has been developed in [16], see also [17].

2.3.1. Construction. Let $N$ be a Lie group which acts properly and freely on the manifold $X$, and put $W = X/N$. We assume that $W$ is simply connected. The normalizer of $N$ in $\text{Diff}(X)$ will be denoted by $\text{Diff}(X,N)$. Let $\pi$ be a group and

$$\rho : \pi \longrightarrow \text{Diff}(X,N)$$

an action of $\pi$ which is properly discontinuous. Put $\pi_N = \rho^{-1}(N)$ for the subgroup of all elements in $\pi$ which act on $X$ via translations of $N \leq \text{Diff}(X)$.

Definition 2.3. Data $(X, N, \pi)$ as above define a Seifert fiber space if the following conditions are satisfied:

1. $\Gamma = \rho(\pi) \cap N$ is a discrete uniform subgroup of $N$,
2. the induced action of $\Theta = \pi/\pi_N$ on $W$ is properly discontinuous.

In the situation of Definition 2.3, $\rho$ is called a Seifert action. If the action of $\pi$ is faithful then it gives rise to an exact sequence of groups

$$1 \longrightarrow \Gamma \longrightarrow \pi \longrightarrow \Theta \longrightarrow 1 ,$$

where $\Gamma = \pi_N$ is isomorphic to a lattice in $N$. Note that, in general, the induced action of $\Theta$ on $W$ may have a finite kernel.

2.3.2. Seifert bundle maps. A Seifert fiber space gives rise to a Seifert bundle map

$$\sigma : X/\pi \longrightarrow W/\pi = W/\Theta$$

whose fibers are compact infra-$N$ spaces. The space $X/\pi$ is called a Seifert bundle over the base orbifold $W/\Theta$ with typical fiber the $N$-manifold $N/\rho(\pi) \cap N$. 
Remark. If $X/\pi$ is a manifold then, in fact, all fibers of $\sigma$ are (compact) infra-$N$ manifolds. If also the base $W/\pi$ is a manifold then all fibers of $\sigma$ are naturally diffeomorphic to $N/\Delta$, where $\Delta \leq \pi$ is the normal subgroup of $\pi$ which acts trivially on $W$. In this situation, since $\pi$ normalizes $N$ and $\Delta$, the Seifert bundle map $\sigma$ defines a fibration whose structure group is discrete and contained in $\text{Aff}(N/\Delta)$. In particular, $\sigma$ is an infra-$N$ bundle with fiber $N/\Delta$ in the sense of Definition 2.2. In general, a Seifert fibering does not give a locally trivial fiber bundle over the orbifold $W/\pi$, but induces the structure of an orbibundle over $W/\pi$. See [17] for various examples.

2.3.3. Seifert structures on manifolds and orbifolds.

Definition 2.4. Let $M$ be a smooth orbifold and $(X,N,\pi)$ a Seifert fiber space. A Seifert fiber structure on $M$ is a diffeomorphism
\[ \varphi : X/\pi \longrightarrow M . \]

2.4. Seifert fiber spaces with nil-geometry. In this subsection let $N$ denote a simply connected nilpotent Lie group. A lattice in $N$ is a discrete uniform subgroup of $N$.

2.4.1. Existence. Let $\Gamma$ be a finitely generated torsion-free nilpotent group and
\[ (1) \quad 1 \longrightarrow \Gamma \longrightarrow \pi \longrightarrow \Theta \longrightarrow 1 \]
an an exact sequence of groups.

Definition 2.5. A Seifert fiber space $(X,N,\pi)$ is said to realize the group extension (1) if $\rho(\Gamma)$ is a lattice of $N$.

Remark. In this situation, $\Gamma$ is a subgroup of finite index in $\pi_N = \rho^{-1}(N)$, and therefore the induced action of $\Theta$ on $W = X/N$ is properly discontinuous.

Theorem 2.6 ([16, §2.2]). Assume that $\Theta$ acts properly discontinuously on a simply connected manifold $W$ and let $N$ be a simply connected nilpotent Lie group which contains $\Gamma$ as a lattice. Then there exists a Seifert fiber space $(X,N,\pi)$, which realizes the group extension (1) and which induces the given action of $\Theta$ on $W$.

2.4.2. Rigidity. Let $N$ and $N'$ be simply connected nilpotent Lie groups, $(X,N,\pi)$ and $(X',N',\pi')$ Seifert fiber spaces, and $\phi : \pi \longrightarrow \pi'$ a homomorphism of groups.

Definition 2.7. An equivariant diffeomorphism of actions
\[ (f,\phi) : (X,\pi) \longrightarrow (X',\pi') \]
is called an affine equivalence of Seifert actions if \( fNf^{-1} = N' \).

Now consider an isomorphism of group extensions of the form

\[
\begin{array}{cccccc}
1 & \longrightarrow & \Gamma & \longrightarrow & \pi & \longrightarrow & \Theta & \longrightarrow & 1 \\
\downarrow & & \downarrow \phi_1 & & \downarrow \phi & & \downarrow \bar{\phi} & & \downarrow \\
1 & \longrightarrow & \Gamma' & \longrightarrow & \pi' & \longrightarrow & \Theta' & \longrightarrow & 1
\end{array}
\]

The Seifert rigidity for nil-geometry may be stated as follows:

**Theorem 2.8 ([16, §2.4]).** Suppose \((X, N, \pi)\) and \((X', N', \pi')\) are Seifert fiber spaces which realize the group extensions in diagram (2). If \((\bar{f}, \bar{\phi}) : (W, \Theta) \to (W', \Theta')\),

is an equivariant diffeomorphism then there exists a lift of \(\bar{f}\) to an affine equivalence of Seifert actions

\[
(f, \phi) : (X, \pi) \to (X', \pi').
\]

Note that, as a consequence, the Seifert rigidity constructs a diffeomorphism of Seifert fiber spaces

\[
X/\pi \to X'/\pi',
\]

which restricts to affine (in particular, connection preserving) maps on the infranil fibers.

### 2.5. Infranil bundles over aspherical manifolds.

Let \(N\) be a simply connected nilpotent Lie group. In this subsection we relate the notions of infra-\(N\) bundle and Seifert fiber space more closely. Recall that a manifold is called aspherical if its universal covering manifold is contractible. The following result asserts that over an aspherical base the structure group of an infra-\(N\) bundle may be reduced to a discrete group. This implies that infranil bundles over an aspherical base are fiberwise affinely equivalent to a Seifert fiber space.

**Proposition 2.9.** Let \(f : M \to B\) be an infra-\(N\) bundle, where \(B\) is an aspherical manifold and \(N\) is a simply connected nilpotent Lie group. Let \(\pi = \pi_1(M)\) and assume also that \(f\) has compact fiber \(N/\Delta\). Then there exists a Seifert fiber space \((X, N, \pi)\) which admits a fiberwise affine diffeomorphism

\[
\varphi : X/\pi \to M
\]

to the infra-\(N\) bundle \(M\).
Proof. By the long exact homotopy sequence of the fibration $f$ we have an exact sequence

$$1 \to \Delta \to \pi_1(M) \to \Theta \to 1$$

where $\Theta \cong \pi_1(B)$. Let $p : W \to B$ be the universal covering of $B$, and $\tilde{f} : M' \to W$ the induced infra-$N$ bundle. The induced morphism of bundles $M' \to M$ is a covering; its group of deck transformations is isomorphic to $\Theta$ and it acts by fiberwise affine maps on $\tilde{f} : M' \to W$.

Since $W$ is contractible, the covering homotopy theorem implies that $\tilde{f} : M' \to W$ is $G$-equivalent to the product bundle $N/\Delta \times W$, where $G = \text{Aff}(N/\Delta)$ is the structure group of $\tilde{f}$. That is, there exists a fiberwise affine diffeomorphism of bundles $N/\Delta \times W \to M'$ over $W$. This map lifts to a diffeomorphism $\tilde{\varphi} : X \to \tilde{M}$ of $X = N \times W$ to the universal cover $\tilde{M}$ of $M$. Via $\tilde{\varphi}$, the group $\pi = \pi_1(M)$ has an induced fiberwise affine action on the (trivial) $N$-principal bundle $X = N \times W$. Let $\Delta_0$ be the maximal nilpotent normal subgroup of $\Delta$. Then $\Delta_0$ is normal in $\pi$, and also $\Delta_0 = \Delta \cap N$ is a lattice in $N$. Since $\pi$ normalizes the lattice $\Delta_0$, the action of $\pi$ on $X$ being fiberwise affine, Lemma 2.10 implies that $\pi$ normalizes $N$, that is, $\pi \leq \text{Diff}(N \times W, N)$. Therefore, the data $(N \times W, N, \pi)$ define a Seifert fiber space which is affinely equivalent to the infra-$N$ bundle $M$. \qed

We remark:

Lemma 2.10. Let $(X, N)$ be a principal bundle and $\Delta_0 \leq N$ a lattice in $N$. Let $N_{\text{Diff}(X)}(\Delta_0)$ denote the normalizer of the left action of $\Delta_0$ in $\text{Diff}(X)$. Then

$$\text{Aff}(X, N) \cap N_{\text{Diff}(X)}(\Delta_0) \leq \text{Diff}(X, N).$$

Proof. Using local trivializations we may assume that $X = N \times W$ is a trivial bundle. Every element $\gamma$ of $\text{Aff}(X, N)$, being a fiberwise affine diffeomorphism of $(X, N)$, gives rise to a family of automorphisms $\ell_w \in \text{Aut}(N)$, $w \in W$, which is defined by the property that

$$\gamma n \gamma^{-1}(v, w) = (\ell_w(n) v, \tilde{\gamma} w),$$

for all $n, v \in N$. (Here $\tilde{\gamma} : W \to W$ denotes the map which $\gamma$ induces on $W$.) If $\gamma$ normalizes $\Delta_0$ then, by Malcev rigidity of the lattice $\Delta_0$, $\ell_w = \ell$ is constant, where $\ell \in \text{Aut}(N)$ is the unique extension of the automorphism which $\gamma$ induces on $\Delta_0$. Therefore $\gamma$ normalizes $N$. \qed
In our context, Seifert fiber spaces may be viewed as infra-N bundles with singular fibers:

**Proposition 2.11.** Let \( f : M_0 \to B \) be an infra-N bundle over the aspherical base manifold \( B \), where \( N \) is a simply connected nilpotent Lie group. Assume that \( f \) has compact fiber \( N/\Delta \) and let \( \mu \leq \text{Aff}(M_0, N/\Delta) \) be a finite group of fiberwise affine diffeomorphisms. Put

\[
M = M_0/\mu.
\]

Let \( \pi = \pi_1(M) \) be the orbifold fundamental group of \( M \) and \( \pi_0 = \pi_1(M_0) \). Then there exists a Seifert fiber space \((X, N, \pi)\) over the base orbifold \( B/\mu \), a Seifert fiber structure \( \varphi : X/\pi \to M \) and a commutative diagram

\[
\begin{array}{ccc}
X/\pi_0 & \xrightarrow{\varphi_0} & M_0 \\
\downarrow & & \downarrow \\
X/\pi & \xrightarrow{\varphi} & M \\
\downarrow & & \downarrow \\
& & B/\mu
\end{array}
\]

where \( \varphi_0 \) is a fiberwise affine diffeomorphism of bundles over \( B \).

**Proof.** Let \( \pi = \pi_1(M) \). By construction, the group \( \pi \) acts as a group of decktransformations of the universal covering \( \tilde{M} \to M \) on \( \tilde{M} \). This extends the corresponding action of \( \pi_0 \), and also induces the action of \( \mu \) on \( M_0 = \tilde{M}/\pi_0 \). Since the fibration \( f \) is \( \mu \)-equivariant, the decktransformation group \( \Delta \leq \pi_0 \) for the fiber of \( f \) is normalized by \( \pi \), and also the maximal nilpotent normal subgroup \( \Delta_0 \) is normalized by \( \pi \). Recall that \( \Delta_0 \) is a lattice in \( N \).

Proposition 2.9 constructs a Seifert fiber space \((X, N, \pi_0)\) and a fiberwise affine diffeomorphism \( \varphi_0 \) as required. Let \( \tilde{\varphi} : (X, N) \to \tilde{M} \) be a lift of the diffeomorphism \( \varphi_0 \) to the universal covers. Then \( \tilde{\varphi} \) is an affine map of bundles. We may pull back the action of \( \pi \) to an action on \( X \). Since by iii) above, \( \mu \) acts on the fibers of \( f \) by affine transformations, the transported action of \( \pi \) on \( X \) is fiberwise affine for the bundle \((X, N)\). That is, \( \pi \) is contained in \( \text{Aff}(X, N) \). Using Lemma 2.10, it follows that, a fortiori, \( \pi \) is contained in \( \text{Diff}(X, N) \). Therefore, the data \((X, N, \pi)\) satisfy the axioms of a Seifert action.

\[\square\]

### 3. Collapsing and Seifert fiberings

We consider a compact aspherical orbifold \( O \), where

\[
O = Y/\Theta,
\]

\( Y \) is a contractible Riemannian manifold and \( \Theta \leq \text{Isom}(Y) \) acts properly discontinuously on \( Y \). Assume further that \((M, g)\) is a closed
Riemannian orbifold which \emph{collapses} to $\mathcal{O}$ with bounded curvatures. This means that there exists a sequence of Riemannian metrics $g_i$ on $M$ with uniformly bounded sectional curvature such that the metric spaces $M_i = (M, g_i)$ converge to $\mathcal{O}$ in the Gromov-Hausdorff distance. That is, we have

$$\lim_{i \to \infty} d_{GH}(M_i, \mathcal{O}) = 0.$$ 

For the Gromov-Hausdorff distance and its generalizations see Appendix A.

In this section, we show that the orbifold $M$ carries a Seifert fiber structure over the base $\mathcal{O}$. This proves Theorem 1.1 in the introduction. In Section 3.3 we also prove Corollary 1.6.

3.1. \textbf{Equivariant collapsing on finite covers.} Let $\Theta_0$ be a torsion-free normal subgroup of $\Theta$, which is of finite index. By our general assumption on orbifolds, such a subgroup of $\Theta$ always exists. Then

$$T = Y/\Theta_0$$

is a closed Riemannian manifold and it is an orbifold covering space of $\mathcal{O}$. Note that $\mathcal{O}$ is a quotient of $T$ by the group of decktransformations of the map $T \to \mathcal{O}$. This is a finite group isomorphic to $\Theta/\Theta_0$.

**Proposition 3.1.** With respect to an appropriate choice of $\Theta_0$, there exists a sequence of Riemannian manifolds $M_i = (\bar{M}, g_i)$, Riemannian orbifold covering maps

$$\bar{M}_i = (\bar{M}, g_i) \longrightarrow (M, g_i),$$

and a finite group $\mu$ which acts by isometries on $\bar{M}_i$ and $T = Y/\Theta_0$ such that

1. $M_i = \bar{M}_i/\mu$ and $\mathcal{O} = T/\mu$;
2. the equivariant Hausdorff distance with respect to $\mu$ satisfies

$$\lim_{i \to \infty} d_{\mu-GH}(\bar{M}_i, T) = 0.$$ 

**Proof.** Let $X_i$ denote the universal cover of $M_i$. Then $M_i = X_i/\pi$, where $\pi$ acts on $X_i$ as a discrete group of isometries. By Proposition A.7 in Appendix A, there exists a sequence $(X_i, \pi, p_i) \in \mathcal{M}^{Groups}$ such that

$$\lim_{i \to \infty} (X_i, \pi, p_i) = (Y, \Theta, q)$$

with respect to the equivariant Gromov-Hausdorff convergence with base point. Note that this implies (compare Definitions A.2, A.3.)

$$\lim_{i \to \infty} (X_i, p_i) = (Y, q).$$
Now let $\pi'$ be a torsion-free finite index normal subgroup of $\pi$. Using Proposition A.4, we infer that there exists a closed subgroup $\Theta' \leq \text{Isom}(Y)$ such that (up to taking a subsequence)

$$\lim_{i \to \infty} (X_i, \pi', p_i) = (Y, \Theta', q).$$

In particular, by Proposition A.6 and taking into account that the involved spaces are compact, we have

$$\lim_{i \to \infty} X_i/\pi' = Y/\Theta'.$$

Let $\mu' = \pi/\pi'$. Then $\mu'$ acts isometrically as a finite group of deck-transformations of the maps $X_i/\pi' \to X_i/\pi$ and furthermore

$$X_i/\pi = (X_i/\pi')/\mu'.$$

Let $\mathcal{M}(\mu')$ denote the space of isometry classes of compact metric spaces with isometric action of $\mu'$. Since $\mu'$ is finite, the equivariant version of the pre-compactness theorem holds with respect to $\mathcal{M}(\mu')$ (compare Proposition A.5 in Appendix A). Therefore, after taking a subsequence, we may assume that there exists

$$\lim_{i \to \infty} (X_i/\pi', \mu') = (\mathcal{O}', \mu') \in \mathcal{M}(\mu').$$

This of course implies that

$$Y/\Theta = \lim_{i \to \infty} X_i/\pi = \lim_{i \to \infty} (X_i/\pi')/\mu' = \mathcal{O'}/\mu'.$$

We deduce, in particular, that $\mathcal{O}' = Y/\Theta'$ is a finite orbifold covering space of $Y/\Theta$ and $\Theta'$ is a normal subgroup of $\Theta$. We have thus constructed a sequence of finite covering manifolds $M_i' = X_i/\pi'$ of $M_i$, which converges to an orbifold covering space $\mathcal{O}' = Y/\Theta'$ of $\mathcal{O}$ in the Gromov-Hausdorff-distance.

Observe that the action of $\pi'$ on $X_i$ is free, since $\pi'$ is torsion-free. Therefore, we may apply Proposition A.8 in Appendix A to a torsion-free finite index normal subgroup $\Theta_0$ of $\Theta'$, and infer that there exists a subgroup $\pi_0$ of $\pi'$ such that

(1) $\pi_0$ is normal and of finite index in $\pi'$, and furthermore

$$\lim_{i \to \infty} (X_i, \pi_0, p_i) = (Y, \Theta_0, q).$$

This gives rise to a sequence of Riemannian manifolds

$$\bar{M}_i = X_i/\pi_0$$

converging to the manifold $T = Y/\Theta_0$ in the Gromov-Hausdorff distance, that is,

$$\lim_{i \to \infty} \bar{M}_i = T.$$
Note that, by passing to a finite index subgroup of \( \pi_0 \), and arguing as in the first part of the proof, we can furthermore assume that

(3) \( \pi_0 \) is normal in \( \pi \).

Now let \( \mu = \pi/\pi_0 \). Then \( \mu \) acts on \( \tilde{M}_i \) by isometries and \( M_i = \tilde{M}_i/\mu \). Furthermore, there exists a sequence \( (\tilde{M}_i, \mu) \in \mathcal{M}(\mu) \) such that

\[
\lim_{i \to \infty} (\tilde{M}_i, \mu) = (T, \mu).
\]

In other words, \( \lim_{i \to \infty} d_{\mu-\text{GH}}(\tilde{M}_i, T) = 0 \). This also implies ordinary Gromov-Hausdorff convergence of \( M_i/\mu \) to \( T/\mu \). Since

\[
\lim_{i \to \infty} \tilde{M}_i/\mu = \lim_{i \to \infty} M_i = \mathcal{O},
\]

we conclude that \( \mathcal{O} = T/\mu \). This proves the proposition. \( \square \)

Proposition 3.1 allows to apply equivariant collapsing results for Riemannian manifolds. Indeed, Theorem B.2 in Appendix B implies that there exist a sequence of \( \mu \)-equivariant fibrations

\[
f_i : \tilde{M}_i \to T
\]

such that, for \( i \) sufficiently large,

i) each fiber \( f_i^{-1}(p) \) carries a flat connection (depending smoothly on \( p \in T \)), such that \( f_i^{-1}(p) \) is affinely diffeomorphic to an infranil manifold \( N_i/\Delta_i \) with canonical flat connection \( \nabla^{\text{can}} \);

ii) the structure group of \( f_i \) is contained in the Lie group

\[
\frac{\mathbb{C}(N_i)}{\mathbb{C}(N_i) \cap \Delta_i} \rtimes \text{Aut}(\Delta_i),
\]

where \( \mathbb{C}(N_i) \) denotes the center of \( N_i \);

iii) The affine structures on the fibers of \( f_i \) are \( \mu \)-invariant;

iv) \( f_i \) is an almost Riemannian submersion.

Note, in particular, that in the terminology of Section 2.3, by ii), \( f_i \) is an infra-\( N_i \) bundle over the base \( T \) with fiber the infranil manifold \( N_i/\Delta_i \), and, by iii), \( \mu \) acts by fiberwise affine maps on the bundle \( f_i \).

3.2. Induced Seifert fiber space structure. Since \( T \) is aspherical, on the level of fundamental groups the long exact homotopy sequence for the fibration \( f_i : \tilde{M}_i \to T \) gives a short exact sequence

(3) \[
1 \longrightarrow \Delta_i \longrightarrow \pi_0 \longrightarrow \Theta_0 \longrightarrow 1,
\]

where \( \Delta_i \cong \pi_1(N_i/\Delta_i) \) is (isomorphic to) a subgroup of \( \text{Aff}(N_i) \). Since the fiber \( N_i/\Delta_i \) of \( f_i \) is infranil, \( \Delta_i \) intersects \( N_i \) in a uniform discrete subgroup \( \Delta_{i0} \), which is of finite index in \( \Delta_i \). In particular, \( \Delta_i \) is a torsion-free finitely generated virtually nilpotent group, and \( \Delta_{i0} \) is its
maximal nilpotent normal subgroup. Since $\Delta_{i0}$ is normal in $\pi_0$, we obtain another short exact sequence
\[
1 \longrightarrow \Delta_{i0} \longrightarrow \pi_0 \longrightarrow \tilde{\Theta}_i \longrightarrow 1,
\]
where $\tilde{\Theta}_i$ acts properly discontinuously on $Y$ with quotient $T$. Proposition 2.9 and its proof now imply:

**Proposition 3.2.** The infra-$N_i$ bundle $f_i : \tilde{M}_i \to T$ together with its flat fiber connections arises from a Seifert fiber construction $(X_i, N_i, \pi_0)$ for the exact sequence (4) over the base $T$.

This means that, for all $i$, there exists a Seifert fiber space $(X_i, N_i, \pi_0)$ over the base $T = Y/\Theta_0$, which realizes the group extension (4), and which has as (so called) typical fiber the nilmanifold $N_i/\Delta_{i0}$. Moreover, there exists a diffeomorphism
\[
\tilde{\varphi}_i : X_i/\pi_0 \to \tilde{M}_i
\]
which induces a commutative diagram
\[
\begin{array}{ccc}
N_i/\Delta_i & \xrightarrow{\sim} & f_i^{-1}(p) \\
\downarrow & & \downarrow \\
X_i/\pi_0 & \xrightarrow{\sim} & \tilde{M}_i \\
\downarrow & & \downarrow \\
Y/\pi_0 & \xrightarrow{\sim} & T \\
\end{array}
\]
\[
\begin{array}{ccc}
& & f_i \\
\downarrow & & \downarrow \\
& & \tilde{M}_i/\mu = M_i \\
\end{array}
\]
such that the fiber maps $N_i/\Delta_i \xrightarrow{\sim} f_i^{-1}(p)$ are affine diffeomorphisms. Since $f_i$ is $\mu$-equivariant, it descends to the projection map $M_i \to \mathcal{O}$.

Let $\tilde{\varphi}_i : X_i \to \tilde{M}_i$ be a lift of $\tilde{\varphi}_i$ to the universal covers. Since by iii) above, $\mu$ acts on the fibers of $f_i$ by affine transformations, the transported action of $\pi$ on $X_i$ is fiberwise affine for the bundle $(X_i, N_i)$. That is, $\pi$ is contained in Aff$(X_i, N_i)$. In the view of Proposition 2.11 we thus have:

**Proposition 3.3.** The Seifert action of $\pi_0$ on $(X_i, N_i)$ extends to a Seifert action of $\pi$ on $(X_i, N_i)$, such that the diffeomorphism $\tilde{\varphi}_i$ descends to a diffeomorphism of orbifolds $\varphi_i : X_i/\pi \to M_i$.

Since $\Delta_i$ is a normal subgroup of $\pi$ and $\Delta_{i0}$ is characteristic in $\Delta_i$, we have induced exact sequences of the form
\[
1 \longrightarrow \Delta_{i0} \longrightarrow \pi \longrightarrow \tilde{\Theta}_i \longrightarrow 1;
\]
moreover, for each \( i \), there is a commutative diagram:

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta_{i0} & \longrightarrow & \pi_0 & \longrightarrow & \bar{\Theta}_0 & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \Delta_{i0} & \longrightarrow & \pi & \longrightarrow & \bar{\Theta}_i & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \Delta & \longrightarrow & \pi & \longrightarrow & \Theta & \longrightarrow & 1 \\
\end{array}
\]

Note that the quotient \( \bar{\Theta}_i \) maps surjectively onto \( \Theta \). Since, \( \Delta_{i0} \) is of finite index in \( \Delta \), \( \bar{\Theta}_i \) acts properly discontinuously on \( Y = X/N_i \). As diagram (5) shows, the quotient space

\[
Y/\bar{\Theta}_i = Y/\Theta
\]

is diffeomorphic to the orbifold \( \mathcal{O} = Y/\Theta \). Therefore, the Seifert action \((\Sigma_i, N_i, \pi)\) realizes the group extension (6) over the orbifold \( \mathcal{O} \). We arrive at:

**Theorem 3.4.** There exists a diffeomorphism of the Seifert fiber space \( X/\pi \) which realizes the group extension (6) over the base \( \mathcal{O} \) to the orbifold \( M_i \).

In particular, the smooth manifold (respectively orbifold) \( M \) carries a Seifert fiber space structure over the orbifold \( \mathcal{O} = Y/\Theta \).

### 3.3. Collapsing of irreducible fake tori.

We show now that irreducible fake smooth tori do not collapse to aspherical Riemannian orbifolds.

**Proposition 3.5.** Let \( \mathcal{T} \) be an irreducible fake smooth torus. Then \( \mathcal{T} \) does not admit a collapse with bounded curvature to a compact aspherical Riemannian orbifold \( \mathcal{O} = Y/\Theta \).

**Proof.** Assume that \( \mathcal{T} \) as above admits such a collapse. By Theorem 1.1, \( \mathcal{T} \) then admits the structure of a Seifert fiber space over \( \mathcal{O} \) with infranil fiber, which realizes a group extension of the form

\[
1 \rightarrow \mathbb{Z}^{n-k} \rightarrow \mathbb{Z}^n \rightarrow \Theta \rightarrow 1 .
\]

Since \( \Theta \) is abelian and it acts properly discontinuously with compact quotient on the contractible manifold \( Y \), we infer that \( \Theta \) is torsion-free abelian, isomorphic to \( \mathbb{Z}^k \), \( k < n \). Therefore, the above extension is isomorphic to the direct product

\[
\mathbb{Z}^n = \mathbb{Z}^{n-k} \times \mathbb{Z}^k .
\]
Note that $\mathcal{O} = Y/\Theta$ is a manifold and a homotopy torus. This implies that $\mathcal{O}$ is also homeomorphic to a torus $T^k$. The Seifert rigidity Theorem 2.8 implies that the Seifert fiber space $\mathcal{T}$ over $\mathcal{O}$ is affinely diffeomorphic to the product $T^{n-k} \times \mathcal{O}$. Therefore, $\mathcal{O}$ is a fake smooth torus. This contradicts the irreducibility of $\mathcal{T}$. □

We can give now the proof for Corollary 1.6:

Proof. Assume to the contrary that $D$-MinVol($\mathcal{T}$) = 0. Since $\mathcal{T}$ is aspherical, this assumption implies (see the proof of Fukaya’s Theorem [9, Theorem 0-1] or [7, Theorem 16.1]) that $\mathcal{T}$ admits a bounded curvature collapse to a compact aspherical Riemannian orbifold $\mathcal{O} = Y/\Theta$. This is not possible by the previous proposition. □

4. SEIFERT FIBERINGS ON INFRASOLV MANIFOLDS

Let $S$ be a simply connected solvable Lie group.

Definition 4.1. A compact infra-$S$ space $S/\Delta$ will be called an infrasolv orbifold. If $S/\Delta$ is a manifold, it is called an infrasolv manifold.

In this section we prove that up to diffeomorphism every infrasolv orbifold arises from a Seifert fiber construction with nil-geometry over a flat Riemannian orbifold. In addition we show that the class of manifolds which are diffeomorphic to infrasolv manifolds is closed with respect to the Seifert fiber construction with nil-geometry. In view of Theorem 1.1, this implies Theorem 1.3 in the introduction.

4.1. Flat orbifolds associated to virtually abelian groups. Let $\Theta$ be a finitely generated virtually abelian group of rank $k$. Then there exists a homomorphism $\Theta \rightarrow \text{Isom}(\mathbb{R}^k)$ such that $\Theta$ acts properly discontinuously on $\mathbb{R}^k$. This homomorphism is unique up to conjugacy by affine transformations (that is, by elements of $\text{Aff}(\mathbb{R}^k)$). (By Bieberbach’s second and third theorems, see, for example, [24].) Let $\mathcal{O}_\Theta = \mathbb{R}^k/\Theta$ denote the flat orbifold associated to $\Theta$.

4.2. Seifert fiberings over flat orbifolds. The following implies Theorem 1.2 in the introduction:

Theorem 4.2. Let $M = S/\pi$ be an infrasolv orbifold, and

$$1 \longrightarrow \Delta_0 \longrightarrow \pi \longrightarrow \tilde{\Theta} \longrightarrow 1$$

an exact sequence of groups, where $\Delta_0$ is a finitely generated torsion-free nilpotent group. Assume further that $\tilde{\Theta}$ is a virtually abelian finitely generated group of rank $k$. Then $M$ admits a Seifert fiber structure over the flat orbifold $\mathcal{O}_\Theta = \mathbb{R}^k/\tilde{\Theta}$, which realizes the group extension (8).
Proof. Since $M$ is diffeomorphic to a standard $\pi$-orbifold $M_\pi$ (see below), it is enough to show the statement for $M_\pi$. Therefore, the theorem is a consequence of Proposition 4.8 below. □

**Corollary 4.3.** Every infrasolv orbifold has a Seifert fiber structure (with nil-geometry) over a flat orbifold.

*Proof.* Indeed, we may consider the Fitting subgroup $\Delta_0$ of $\pi$, which is (by definition) the maximal nilpotent normal subgroup of $\pi$. This gives rise to an exact sequence of the form (8). □

Let $\pi$ be a torsion-free virtually polycyclic group. Then (by [2], but see also Section 4.3.4 below), there exists an infrasolv manifold $M_\pi$ with fundamental group $\pi$.

**Corollary 4.4.** Let $X/\pi$ be the total space of a Seifert fiber space with nil-geometry over a flat orbifold $O = \mathbb{R}^k/\Theta$. Then $X/\pi$ is (diffeomorphic to) an infrasolv manifold.

*Proof.* Since $X/\pi$ is a Seifert fiber space over the base $O$, there exists a homomorphism $\pi \to \Theta$, as in (8). Let $M_\pi$ be an infrasolv manifold with fundamental group $\pi$, and $f : M_\pi \to O$ the bundle map for the Seifert fiber structure on $M_\pi$ which realizes the group extension representing the map $\pi \to \Theta$. By Theorem 2.8 (uniqueness of the Seifert construction), the infra-$N$ bundles $M_\pi$ and $X/\pi$ are fiberwise affinely diffeomorphic over $O$. In particular, $M_\pi$ and $X/\pi$ are diffeomorphic. □

### 4.3. Flat affine manifolds

Let $U$ be a unipotent real linear algebraic group. Then $U$ is in particular a simply connected nilpotent Lie group. Conversely, every simply connected nilpotent Lie group admits the structure of a unipotent real linear algebraic group. (Cf. [19, Preliminaries, §2] for basic facts on linear algebraic groups.)

**4.3.1. Flat manifolds modeled on $U$.** We consider manifolds which carry locally the affine geometry of $U$. This generalizes the notion of infra-$U$ manifold, which is an essentially metric concept.

Let $\text{Aff}(U) = \text{Aff}(U, \nabla^{\text{can}})$ be the affine group of $U$.

**Definition 4.5.** Let $\Delta \leq \text{Aff}(U)$ be a discrete subgroup which acts properly on $U$ with compact quotient. Then the quotient

$$U/\Delta$$

is called a flat affine $U$-orbifold. If $\Delta$ acts freely on $U$ then $U/\Delta$ is a flat affine $U$-manifold.

*Remark.* A long standing conjecture ("Auslander’s conjecture") asserts that $\Delta$ must be a virtually polycyclic group.
4.3.2. **Unipotent shadow and algebraic hull.** Now let $\Delta$ be a virtually polycyclic group without a finite (non-trivial) normal subgroup. By a construction, originally due to Mostow and Auslander (see [19, Chapter IV, Proposition 4.4] and [1, Appendix A.1], for reference), we may associate to $\Delta$ in a unique way a unipotent real linear algebraic group $U_\Delta$, which is called the **unipotent shadow** of $\Delta$. The unipotent shadow $U_\Delta$ has the following characterizing additional properties:

1. $\dim U_\Delta = \text{rank } \Delta$. (Recall that the rank of $\Delta$ coincides with the virtual cohomological dimension of $\Delta$.)
2. There exists a $\mathbb{Q}$-defined real linear algebraic group $H = H_\Delta$ which has $\Delta \leq H(\mathbb{Q})$ as a Zariski-dense subgroup.
3. There is a splitting of algebraic groups $H = U_\Delta \rtimes T$, such that $T$ is a subgroup of semi-simple elements which acts faithfully on $U_\Delta$.

The group $H$ is called the **algebraic hull** of $\Delta$.

4.3.3. **Standard $\Delta$-manifolds.** Since $T$ acts faithfully on $U_\Delta$, we may realize the hull $H = H_\Delta$ as a subgroup of $\text{Aff}(U_\Delta)$ in a natural way. In particular, in this way, $\Delta$ acts faithfully on $U_\Delta$ by affine transformations. Then we have:

**Theorem 4.6** (cf. [1, Theorem 1.2]). The affine action of $\Delta$ on $U_\Delta$ is properly discontinuous, and it is uniquely defined up to conjugacy with an element of the affine group $\text{Aff}(U)$.

The quotient space

$$M_\Delta = U_\Delta / \Delta$$

is called a **standard $\Delta$-orbifold**. If $M_\Delta$ is a manifold it is called a **standard $\Delta$-manifold**. Remarkably, in standard $\Delta$-manifolds all homotopy equivalences of maps are induced by affine automorphisms of $U_\Delta$, see [1].

**Remark.** If $\bar{\Delta}$ is an arbitrary virtually polycyclic group, dividing by its maximal finite normal subgroup yields a unique homomorphism $\bar{\Delta} \to \Delta$ onto the fundamental group of a standard $\Delta$-orbifold. We also call $M_{\bar{\Delta}} = M_\Delta$ the standard $\Delta$-orbifold associated to $\Delta$.

4.3.4. **Smooth rigidity of infrasolv manifolds.** As remarked in [1], every standard $\Delta$-manifold admits a (in general, non-unique) infrasolv manifold structure modeled on some solvable Lie group. In particular, every torsion-free group $\Delta$ as above appears as the fundamental group of an infrasolv manifold. Moreover, standard $\Delta$-manifolds may serve as **unique** smooth models for infrasolv manifolds:
Theorem 4.7 (cf. [1, Theorem 1.4]). Let \( M = S/\Delta \) be a (compact) infrasolv manifold, where \( S \) is a simply connected solvable Lie group and \( \Delta \leq \text{Aff}(S) \) a discrete group of isometries. Then \( M \) is smoothly diffeomorphic to the standard \( \Delta \)-manifold \( M_\Delta \).

The corresponding statements for infrasolv orbifolds and standard \( \Delta \)-orbifolds hold as well, with the same proof as in [1].

4.4. Seifert fiberings on standard-manifolds. Let \( M_\pi \) be a standard \( \pi \)-orbifold, where \( \pi \) is a virtually polycyclic group. Consider an exact sequence of groups
\[
1 \rightarrow \Delta_0 \rightarrow \pi \rightarrow \bar{\Theta} \rightarrow 1,
\]
where \( \Delta_0 \) is a finitely generated torsion-free nilpotent group.

Proposition 4.8. There exists a Seifert fiber structure for \( M_\pi \), which realizes the group extension (9) over the standard \( \bar{\Theta} \)-orbifold \( M_{\bar{\Theta}} \).

Proof. Let \( U = U_\pi \) be the unipotent shadow of \( \pi \). Recall that for the standard affine embedding \( \pi \leq \text{Aff}(U) \) the intersection \( U \cap \pi \) is the maximal nilpotent normal subgroup \( \text{Fitt}(\pi) \) of \( \pi \) (see [1]). Since \( \Delta_0 \) is nilpotent, we therefore have \( \Delta_0 \leq U \). Let \( U_0 \) be the Zariski-closure of \( \Delta_0 \) in \( U \). Then \( \Delta_0 \) is a finite index subgroup of \( \Delta' = U_0 \cap \pi \). Moreover, \( U_0 \) is normalized by \( \pi \), since \( \Delta_0 \) is a normal subgroup of \( \pi \). Since \( \pi \) is Zariski-dense in \( H = \text{UT} \) it follows, in particular, that \( U_0 \) is normal in \( H \) and hence also in \( U \). Let \( U_1 = U/U_0 \) be the quotient unipotent \( \mathbb{Q} \)-defined real linear algebraic group.

We consider now the induced homomorphism \( H \rightarrow \text{Aff}(U_1) \). Let \( \Theta \) denote the image of \( \pi \) in \( \text{Aff}(U_1) \). It clearly is a Zariski dense subgroup of the image \( H_1 \) of \( H \) in \( \text{Aff}(U_1) \). Since \( U_1 \) is the unipotent radical of \( H_1 \), we have the relation \( \dim U_0 = \dim U_1 \leq \text{rank } \Theta \) (see [19, Chapter IV, Lemma 4.36]). Since \( \Delta_0 \) is contained in the kernel of \( \pi \rightarrow \Theta \), we have \( \text{rank } \Theta \leq \text{rank } \pi - \text{rank } \Delta_0 = \dim U - \dim U_0 \). We conclude that \( \text{rank } \Theta = \dim U_1 \). It is now evident that the image \( H_1 \) of \( H \) in \( \text{Aff}(U_1) \) satisfies the axioms for an algebraic hull of \( \Theta \). Therefore \( M_{\Theta} = U_1/\Theta \) is a standard \( \Theta \)-manifold. The data \((X, N, \pi) = (U, U_0, \pi)\) define a Seifert bundle map
\[
\sigma : M_\pi \rightarrow M_{\Theta}
\]
and the Seifert construction \((U, U_0, \pi)\) realizes the group extension (9) over \( M_{\Theta} = M_\Theta \).

Corollary 4.9. The class of smooth orbifolds represented by standard \( \Delta \)-orbifolds is closed with respect to the Seifert fiber construction with nil-geometry.
**Proof.** Indeed, let \( \sigma : M \rightarrow M_\Theta \) be a Seifert fiber space over a standard orbifold \( M_\Theta \), which realizes a group extension of the form (9). By the previous proposition, the standard-\( \pi \) orbifold \( M_\pi \) also supports a Seifert construction with nil-geometry over \( M_\Theta \), which realizes (9). By the Seifert rigidity theorem (Theorem 2.8), \( M \) and \( M_\pi \) are diffeomorphic. \( \square \)

5. APPENDIX

In the following sections we gather several important notions and results from equivariant Gromov-Hausdorff convergence, developed by Fukaya and Yamaguchi, and the work of Cheeger, Fukaya, and Gromov on collapsed Riemannian manifolds with bounded sectional curvature. General references for these topics are [7] and [15].

**Appendix A. (Equivariant) Gromov-Hausdorff Convergence**

Let us first recall the definition of the classical Hausdorff distance: For subsets \( A \) and \( B \) of a metric space \( X \), the **Hausdorff distance** between \( A \) and \( B \) in \( X \), \( d^X_H(A,B) \), is defined as the infimum of all positive real numbers \( \varepsilon \) such that the open \( \varepsilon \)-neighbourhood of \( A \) in \( X \) contains \( B \) and vice versa. For any metric space \( X \), the Hausdorff distance \( d^X_H \) then defines a metric on the set of all closed and bounded nonempty subsets of \( X \).

Around 1980, Gromov introduced an abstract version of the Hausdorff distance as follows: The **Gromov-Hausdorff distance** of two compact metric spaces \( X \) and \( Y \), \( d_{GH}(X,Y) \), is defined as the infimum of all numbers \( d^Z_H(f(X),g(Y)) \), where \( Z \) ranges over all metric spaces in which \( X \) and \( Y \) can be imbedded isometrically, and \( f \) and \( g \) over all isometric embeddings \( X \rightarrow Z \) and \( Y \rightarrow Z \). Alternatively, the Gromov-Hausdorff distance of \( X \) and \( Y \) can also be defined as the infimum of all Hausdorff distances \( d^Z_{GH}(X,Y) \), where \( Z \) is the disjoint union of \( X \) and \( Y \) and where the infimum is now taken over all metrics on \( Z \) which extend the metrics on \( X \) and \( Y \).

The Gromov-Hausdorff distance \( d_{GH} \) defines a metric on the collection \( \mathcal{C} \) of all compact metric spaces, considered up to isometry, and therefore gives rise to the notion of **Gromov-Hausdorff convergence** of sequences of compact metric spaces. Moreover, with respect to \( d_{GH} \), the space \( \mathcal{C} \) is complete.
It is sometimes also convenient to work with so-called (Gromov-) Hausdorff approximations: If $X$ and $Y$ are compact metric spaces and $\varepsilon$ is real and positive, a (not necessarily continuous) map $\phi : X \to Y$ is called an $\varepsilon$–Hausdorff approximation from $X$ to $Y$ if $|d_Y(\phi(x), \phi(x')) - d_X(x, x')| \leq \varepsilon$ for all $x, x' \in X$ and if the $\varepsilon$–neighbourhood of $\phi(X)$ in $Y$ is equal to $Y$.

Notice that if $d_{\text{GH}}(X, Y)$ is less than $\varepsilon$, then there exist $3\varepsilon$–Hausdorff approximations from $X$ to $Y$ and $Y$ to $X$. Conversely, if there exist $\varepsilon$–Hausdorff approximations between $X$ and $Y$, then $d_{\text{GH}}(X, Y) \leq 3\varepsilon$. In particular, Gromov-Hausdorff convergence can therefore also be defined via Hausdorff approximations.

There is also a useful notion of Gromov-Hausdorff convergence for noncompact metric spaces, at least for length spaces in which, by definition, the distance between points is given by the infimum of the lengths of all curves joining the points:

A sequence $(X_n, x_n)_{n \in \mathbb{N}}$ of locally compact length spaces with given basepoints $x_n \in X_n$ is said to converge to a pointed metric space $(Y, y)$ in the pointed Gromov-Hausdorff sense if, for all $r > 0$, the closed $r$-balls $B_r(x_n)$ Gromov-Hausdorff converge in the usual sense to the closed $r$-ball $B_r(y)$ in $Y$.

Under this so-called pointed Gromov-Hausdorff convergence or Gromov-Hausdorff convergence with basepoint, the collection $\mathcal{M}$ of all locally compact complete pointed length spaces, considered up to isometries that preserve basepoints, is complete as well. Moreover, the same is true for ordinary Gromov-Hausdorff convergence in the subspaces $\mathcal{L}(D)$ of $\mathcal{C}$ which are made up of the isometry classes of compact length spaces subject to a given uniform upper bound $D$ for the diameter.

In the context of Riemannian manifolds with lower curvature bounds Gromov obtained the following fundamental result:

**Theorem A.1 (Gromov’s Precompactness Theorem)** For any natural number $m$ and any real numbers $\kappa$ and $D$, the class of closed Riemannian $m$-manifolds with Ricci curvature $\text{Ric} \geq (m - 1)\kappa$ and diameter $\text{diam} \leq D$ is precompact in $\mathcal{L}(D)$ with respect to Gromov-Hausdorff convergence, and the class of pointed complete Riemannian $m$-manifolds with $\text{Ric} \geq (m - 1)\kappa$ is precompact in $\mathcal{M}$ with respect to the pointed Gromov-Hausdorff topology.

If one replaces the Ricci curvature bounds in Theorem A.1 by sectional curvature ones, one may actually replace $\mathcal{L}(D)$ and $\mathcal{M}$ by appropriate classes of Alexandrov spaces, but we will not need this here.
Instead, let us now turn to notions of Gromov-Hausdorff convergence for metric spaces which are equipped with isometric group actions.

Let $\mathcal{M}$ be as above, and let $\mathcal{M}^{\text{Groups}}$ consist of all triples $(X, \Gamma, p)$ with $(X, p) \in \mathcal{M}$ such that $\Gamma$ is a closed subgroup of the isometry group of $X$, and let in this case for real positive $r$ be $\Gamma(r)$ the set of all $\gamma \in \Gamma$ for which $d(\gamma p, p) < r$.

**Definition A.2** An $\varepsilon$-equivariant Hausdorff approximation with basepoint $p$ from $(X, \Gamma, p) \in \mathcal{M}^{G}$ to $(Y, \Lambda, q) \in \mathcal{M}^{\text{Groups}}$ is given by a triple of maps $(f, \phi, \psi)$ with

$$
\begin{align*}
f : B_p(1/\varepsilon, X) &\rightarrow Y, \\
\phi : \Gamma(1/\varepsilon) &\rightarrow \Lambda(1/\varepsilon) \quad \text{and} \\
\psi : \Lambda(1/\varepsilon) &\rightarrow \Gamma(1/\varepsilon)
\end{align*}
$$

that satisfies the following conditions:

1. $f(p) = q$;

2. the $\varepsilon$-neighbourhood of $f(B_p(1/\varepsilon, X))$ contains $B_q(1/\varepsilon, Y)$;

3. for all $x_1, x_2 \in B_p(1/\varepsilon, X)$ one has

$$
| d(f(x_1), f(x_2)) - d(x_1, x_2) | < \varepsilon;
$$

4. if $\gamma \in \Gamma(1/\varepsilon)$, $x \in B_p(1/\varepsilon, X)$, and $\gamma x \in B_p(1/\varepsilon, X)$, then

$$
d(f(\gamma x), \phi(\gamma)(f(x))) < \varepsilon;
$$

5. if $\mu \in \Lambda(1/\varepsilon)$, $x \in (B_p(1/\varepsilon, X)$, and $\psi(\mu)(x) \in (B_p(1/\varepsilon, X)$, then

$$
d(f(\psi(\mu)(x)), \mu(f(x)) < \varepsilon.
$$

Notice that it is neither required that the maps occuring here be globally defined, nor that they be continuous, nor, as regards $\phi$ and $\psi$, be homomorphisms.

**Definition A.3** Let $(X, \Gamma, p), (Y, \Lambda, q) \in \mathcal{M}^{\text{Groups}}$. The equivariant Gromov-Hausdorff distance with basepoint $d_{GH}((X, \Gamma, p), (Y, \Lambda, q))$ is defined as the infimum of all positive real numbers $\varepsilon$ such that there exist $\varepsilon$-equivariant Hausdorff approximations with basepoint from $(X, \Gamma, p)$ to $(Y, \Lambda, q)$ and vice versa.
If \((X_n, \Gamma_n, p_n), (Y, \Lambda, q) \in M^\text{Groups}\), the equation

\[
\lim_{n \to \infty} (X_n, \Gamma_n, p_n) = (Y, \Lambda, q)
\]

means that

\[
\lim_{n \to \infty} d_GH((X_n, \Gamma_n, p_n), (Y, \Lambda, q)) = 0.
\]

If the groups that occur in the above definition are isomorphic to a
fixed group \(G\), we will also speak of \(G\)-equivariant Gromov-Hausdorff
distance (with basepoint).

In the case where all groups are trivial, we will write \((X, p)\) instead
of \((X, \text{id}, p)\) etc. and \(d_GH\) instead of \(d_G^{GH}\), and equivariant Gromov-
Hausdorff convergence with basepoint then reduces to ordinary Gromov-
Hausdorff convergence with basepoint.

Let us also note that in the situation where all spaces in question are
compact and have uniformly bounded diameters, Gromov-Hausdorff
convergence and Gromov-Hausdorff convergence with base point are
essentially equivalent concepts: The latter clearly implies the former,
and up to passing to a subsequence the converse also holds.

Let us now gather several important facts and results concerning
equivariant Gromov-Hausdorff convergence.

**Proposition A.4** ([13]) If \((X_n, \Gamma_n, p_n) \in M^\text{Groups} and \((Y, q) \in M\)
satisfy with respect to the Gromov-Hausdorff distance with basepoint

\[
\lim_{n \to \infty} (X_n, p_n) = (Y, q),
\]

then there exist \(\Lambda\) and a subsequence \(\{n_k\}_{k \in \mathbb{N}}\) such that \((Y, \Lambda, q) \in M^\text{Groups}\) and such that

\[
\lim_{k \to \infty} (X_{n_k}, \Gamma_{n_k}, p_{n_k}) = (Y, \Lambda, q).
\]

Notice that even if all \(\Gamma_n\) and \(X_n\) above are the same, in general the
limit group \(\Lambda\) can be a different group. However, for compact groups
and spaces we have the following stability result.

**Proposition A.5** ([7, Theorem 6.9]) Let \(G\) be a group and \(M(G)\)
denote the set of all isometry classes of compact metric spaces equipped
with an isometric \(G\)-action. If \(G\) is compact, then for any natural num-
ber \(m\) and any real numbers \(\kappa\) and \(D\), the class of closed \(m\)-dimensional
Riemannian \(G\)-manifolds with Ricci curvature \(\text{Ric} \geq (m - 1) \kappa\) and
diameter diam $\leq D$ is precompact in $\mathcal{M}(G)$ with respect to the $G$-equivariant Gromov-Hausdorff topology.

The following results establish several relations between equivariant Gromov-Hausdorff convergence and convergence of subgroups and quotient spaces.

**Proposition A.6 ([12])** If $\left( X_n, \Gamma_n, p_n \right), \left( Y, \Lambda, q \right) \in \mathcal{M}^{\text{Groups}}$ and

$$\lim_{n \to \infty} \left( X_n, \Gamma_n, p_n \right) = \left( Y, \Lambda, q \right),$$

then

$$\lim_{n \to \infty} \left( X_n/\Gamma_n, \bar{p}_n \right) = \left( Y/\Lambda, \bar{q} \right),$$

where $\bar{p}_n$ and $\bar{q}$ denote the images of $p_n$ and $q$, respectively, under the canonical orbit space projections $X_n \to X_n/\Gamma_n$ and $Y \to Y/\Lambda$.

**Proposition A.7 ([12])** Let $X_n$ and $Y$ be complete and simply connected Riemannian manifolds and $\Gamma_n$ resp. $\Lambda$ be closed subgroups of the isometry groups of $X_n$ resp. $Y$ which all act effectively and properly discontinuously. Suppose, moreover, that the sectional curvatures of $X_n$ and $Y$ are uniformly bounded in absolute value. Then

$$\lim_{n \to \infty} \left( X_n/\Gamma_n, \bar{p}_n \right) = \left( Y/\Lambda, \bar{q} \right)$$

implies

$$\lim_{n \to \infty} \left( X_n, \Gamma_n, p_n \right) = \left( Y, \Lambda, q \right).$$

**Proposition A.8 ([13])** Let $\left( X_n, \Gamma_n, p_n \right), \left( Y, \Lambda, q \right) \in \mathcal{M}^{\text{Groups}}$ satisfy

$$\lim_{n \to \infty} \left( X_n, \Gamma_n, p_n \right) = \left( Y, \Lambda, q \right)$$

and let $\Lambda'$ be a normal subgroup of $\Lambda$ such that the following conditions hold:

1. $\Lambda'/\Lambda$ is discrete and finitely presented;
2. $Y/\Lambda$ is compact;
3. there exists $r > 0$ such that $\Lambda'$ is generated by $\Lambda'(r)$ and such that the homomorphism $\pi_1(B_q(r, Y), q) \to \pi_1(Y, q)$ is surjective;
4. for all $n$, $X_n$ is simply connected and $\Gamma_n$ acts freely and properly discontinuously on $X_n$.

Then there exists a sequence of normal subgroups $\Gamma'_n$ of $\Gamma_n$ satisfying

(a) $\lim_{n \to \infty} \left( X_n, \Gamma'_n, p_n \right) = \left( Y, \Lambda', q \right);$. 
(b) $\Gamma_n/\Gamma'_n$ is isomorphic to $\Lambda/\Lambda'$ for all sufficiently large $n$.

**Appendix B. Collapsing of manifolds under both-sided bounds on sectional curvature**

We will now briefly describe the collapsing of Riemannian manifolds with both-sided bounds on sectional curvature, where a detailed structure theory has been initiated and developed in the works of Cheeger, Fukaya, Gromov, and others.

Let $M = (M^m, g)$ be a Riemannian manifold of dimension $m$ and let $FM = F(M^m)$ denote its bundle of orthonormal frames. When fixing a bi-invariant metric on $O(m)$, the Levi-Civita connection of $g$ gives rise to a canonical metric on $FM$, so that the projection $FM \to M$ becomes a Riemannian submersion and so that $O(m)$ acts on $FM$ by isometries. Another fibration structure on $FM$ is called $O(m)$ invariant if the $O(m)$ action on $FM$ preserves both its fibres and its structure group.

A pure $N$-structure on $M^m$ is defined by an $O(m)$ invariant fibration, $\tilde{\eta} : FM \to B$, with fibre a nilmanifold isomorphic to $(N/\Gamma, \nabla^{\text{can}})$ and structural group contained in the group of affine automorphisms of the fibre, where $B$ is a smooth manifold, $N$ is a simply connected nilpotent group and $\nabla^{\text{can}}$ the canonical connection on $N$ for which all left invariant vector fields are parallel. A pure $N$-structure on $M$ induces, by $O(m)$-invariance, a partition of $M$ into “orbits” of this structure, and is then said to have positive rank if all these orbits have positive dimension. A pure $N$-structure $\tilde{\eta} : FM \to B$ over a Riemannian manifold $(M, g)$ gives rise to a sheaf on $FM$ whose local sections restrict to local right invariant vector fields on the fibres of $\tilde{\eta}$, and if the local sections of this sheaf are local Killing fields for the metric $g$, then $g$ is said to be invariant for the $N$-structure (and $\tilde{\eta}$ is then also sometimes referred to as pure nilpotent Killing structure for $g$). Cheeger, Fukaya and Gromov showed:

**Theorem B.1** ([4]) Let for $m \geq 2$ and $D > 0$ $\mathfrak{M}(m, D)$ denote the class of all $m$-dimensional compact connected Riemannian manifolds $(M, g)$ with sectional curvature $|K_g| \leq 1$ and diameter $\text{diam}(g) \leq D$. 
Then, given any $\varepsilon > 0$, there exists a positive number $v = v(m, D, \varepsilon) > 0$ such that if $(M, g) \in \mathcal{M}(m, D)$ satisfies $\text{vol}(g) < v$, then $M^n$ admits a pure $N$-structure $\tilde{\eta} : FM \to B$ of positive rank so that

(a) there is a smooth metric $g_\varepsilon$ on $M$ which is invariant for the $N$-structure $\tilde{\eta}$ and for which all fibres of $\tilde{\eta}$ have diameter less than $\varepsilon$, satisfying

$$e^{-\varepsilon} g < g_\varepsilon < e^\varepsilon g, \quad |\nabla_g - \nabla_{g_\varepsilon}| < \varepsilon, \quad |\nabla_{g_\varepsilon} R_{g_\varepsilon}| < C(m, l, \varepsilon);$$

(b) there exist constants $i = i(m, \varepsilon) > 0$ and $C = C(m, \varepsilon)$ such that, when equipped with the metric induced by $g_\varepsilon$, the injectivity radius of $B$ is $\geq i$ and the second fundamental form of all fibres of $\tilde{\eta}$ is bounded by $C$.

The $O(m)$ invariance of a pure $N$-structure $\tilde{\eta} : FM \to B$ implies that the $O(m)$ action on $FM$ descends to an $O(m)$ action on $B$ and that the fibration on $FM$ descends to a possibly singular fibration on $M$, $\eta : M^n \to B/O(m)$, such that the following diagram commutes.

$$\begin{array}{ccc}
F(M^n) & \xrightarrow{\tilde{\eta}} & B \\
\downarrow \pi & & \downarrow \pi \\
M^n & \xrightarrow{\eta} & B/O(m)
\end{array}$$

Theorem B.1 describes the structure of a general collapse with bounded curvature, where $B/O(m)$ is in general not a manifold. However, in the case where one has a bounded curvature collapse to a manifold limit space, there is also Fukaya’s following equivariant fibre bundle version of the above result:

**Theorem B.2** ([7, 8, 10]) Let $M_i \in \mathcal{M}(m, D)$, $N \in \mathcal{M}(n, D)$ and let $G$ be a compact Lie group which acts on $M_i$ and $N$ by isometries so that the $M_i$ converge for $i \to \infty$ to $N$ in the $G$-equivariant Gromov-Hausdorff distance. Then, for $i$ sufficiently large, there exist mappings $\pi_i : M_i \to N$ which satisfy

(a) $\pi_i : M_i \to N$ is a fibre bundle and $\pi_i$ is $G$-equivariant;

(b) each fiber $\pi_i^{-1}(p)$ carries a flat connection which depends smoothly on $p \in N$, such that $\pi_i^{-1}(p)$ is affinely diffeomorphic to an infranil manifold $N_i/\Delta_i$ with its canonical flat connection $\nabla_{\text{can}}$. 

(c) the affine structures on the fibres of $\pi_i : M_i \to N$ are $G$-invariant;

(d) the structure group of $\pi_i$ is contained in the Lie group
\[ \frac{C(N_i)}{C(N_i) \cap \Delta_i} \rtimes \text{Aut}(\Delta_i), \]
where $C(N_i)$ denotes the center of $N_i$;

(e) $\pi_i$ is an almost Riemannian submersion in the sense that for any $p \in N$ and any tangent vector $v \in T_p M_i$ orthogonal to the fibre $\pi_i^{-1}(p)$, one has an estimate
\[ e^{-o(i)} < \frac{||d\pi_i(v)||}{||v||} < e^{o(i)}, \]
where $o(i)$ satisfies $\lim_{i \to \infty} o(i) = 0$.

References


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