Exercise 51: Prove the following subtasks using the mean value theorem:

(a) Show the inequality
\[ \ln(1 + x) \leq \frac{x}{\sqrt{1 + x}} \quad \text{for } x > 0. \]

Hint: Consider the function \( f(t) = \ln(1 + t) - \frac{t}{\sqrt{1 + t}} \) in the interval \([0, x]\).

(b) Show the estimates
\[ 1 - \frac{1}{x} < \ln x < x - 1, \quad x \in (1, \infty). \]

Can you give an upper and lower bound to the real number \( a = 2 \ln 3 - 3 \ln 2 \) using these inequalities?

Hint: Consider the function \( f(t) = \ln t \) for \( t \in (1, x) \).

Exercise 52: Compute the limits

(a) \( \lim_{x \to 0} \frac{\cos x + 3x - 1}{2x} \),
(b) \( \lim_{x \to a} \frac{x^a - a^x}{a^x - a^x} \), \( a > 0, a \neq 1 \),
(c) \( \lim_{x \to \infty} \frac{2 \ln x}{x^b} \), \( b > 0 \),
(d) \( \lim_{x \to \infty} \frac{\ln(\ln x)}{\ln x} \).

Exercise 53: Determine \( c \in \mathbb{R} \), such that the function \( f: \mathbb{R}_{>0} \to \mathbb{R} \) given by
\[ f(x) = \begin{cases} 
1 - \frac{1}{\ln x} & \text{for } x \neq 1, \\
\frac{1}{x - 1} & \text{for } x = 1,
\end{cases} \]
is continuous.

Exercise 54: Consider the function \( f: [-1, 2) \cup (2, 3] \to \mathbb{R} \) with
\[ f(x) = \frac{x^3 - 4x^2 + x + 6}{x - 2}. \]

Show that \( f(x) \leq 0 \) is satisfied for all \( x \in [-1, 2) \cup (2, 3] \).

Exercise 55: The suburban train S3 runs with a maximum speed of 160 km/h between Karlsruhe and Bruchsal. The power consumption of such a train is proportional to the square of its velocity. Powering the train at a speed of 50 km/h costs 100 EUR per hour. There are additional costs of 400 EUR per hour (labour costs, maintenance costs, etc.).

(a) At which speed are the operation costs per kilometer minimal?

(b) The mean daily revenue is 14 EUR per kilometer. How fast should the train go to maximize the total profit (i.e., the difference between revenue and costs) per hour?

Due date: Please hand in your homework on Monday, January 21, 12:00, into the AM1-box in the student office in the International Department.
Exercise T32: Use the mean value theorem to prove
(a) the inequality
\[ |\cos e^x - \cos e^y| \leq |x - y| \]
for \( x, y \leq 0 \),
(b) the limit
\[ \lim_{n \to \infty} (\sqrt[n]{n^2 + k^2} - \sqrt[n]{n^2}) = 0. \]
Hint: Consider the function \( x \mapsto \sqrt[n]{x} \).

Exercise T33: Compute the given limits. Why is it not allowed to use de l’Hôpital’s rule in part (c)?
(a) \[ \lim_{x \to 0} \frac{\tan x - x}{x^3} \]
(b) \[ \lim_{x \to \infty} \frac{x^2 e^x}{(e^x - 1)^2} \]
(c) \[ \lim_{x \to \infty} \frac{x - \sin x}{x + \sin x} \]

Exercise T34: Show that for \( x \in [-1, 1] \)
\[ -1 + x - \frac{x^2 - 1}{x^2 + 1} \leq \arctan(x) \leq 1 + x + \frac{x^2 - 1}{x^2 + 1}. \]

Hint: Analyze the extrema of the differences of the terms.

For detailed information regarding this course please check the page
http://www.math.kit.edu/iag6/lehre/am12012w/en

Tutorial date: Wednesday, January 16, 2013, 14:00-15:30 pm.