Exercise 6: Determine all solutions of the following systems of linear equations:

(a) \[
\begin{align*}
-6x_1 - 9x_2 + x_3 &= -8 \\
-6x_1 - 7x_2 - x_3 &= -4 \\
-3x_1 + 4x_2 - 3x_3 &= -5
\end{align*}
\]

(b) \[
\begin{align*}
2x_1 + x_2 + 4x_3 + 3x_4 &= 0 \\
-x_1 + 2x_2 + x_3 - x_4 &= 4 \\
3x_1 + 4x_2 - x_3 - 2x_4 &= 0 \\
4x_1 + 3x_2 + 2x_3 + x_4 &= 0
\end{align*}
\]

(c) \[
\begin{align*}
3x_1 - 2x_2 + 3x_3 &= 7 \\
-2x_1 + 4x_2 - 2x_3 &= -1
\end{align*}
\]

Exercise 7: Determine all values of \( \alpha \in \mathbb{R} \) for which the following system has at least one solution.

\[
\begin{align*}
\alpha^2 x + (2\alpha^2 - 4)y + (2\alpha^2 + 1)z &= \alpha - 10 \\
2x + y + 5z &= -6 \\
\alpha^2 x + (2\alpha^2 + 1)y + 2\alpha^2 z &= \alpha + 2
\end{align*}
\]

Exercise 8: Let \( n \in \mathbb{N} \) be a natural number.

a) Use mathematical induction to show

\[
\sum_{k=1}^{n} k^3 = \left[ \frac{n(n+1)}{2} \right]^2.
\]

b) Show the identity

\[
\sum_{k=1}^{n} k^3 = \left( \sum_{k=1}^{n} k \right)^2.
\]

Hint: The fist lecture could be helpful for the b)-part.

Exercise 9: Let \( n \geq 4 \) be a natural number. Use mathematical induction to show

\[
2^n \geq n^2.
\]

Exercise 10: Prove the following statement:

\[
\sum_{k=1}^{n} (-1)^{k-1}k^2 = (-1)^{n-1} \binom{n+1}{2}, \quad n \in \mathbb{N}.
\]
Exercise T4:
(a) Determine all solutions of the system of linear equations
\[
\begin{align*}
  x_1 - x_2 + x_3 + x_4 &= 0 \\
  2x_1 + x_2 - x_3 + 2x_4 &= 0 \\
  3x_1 + 2x_2 + x_3 &= 3
\end{align*}
\]
(b) Determine all real numbers \(\alpha\) and \(\beta\), such that the following system of linear equations has (i) a unique solution, (ii) more than one solution and (iii) no solution?
\[
\begin{align*}
  \alpha x_1 + x_2 + 2x_3 &= 1 \\
  -x_1 + 3x_2 + x_3 &= \beta \\
  2x_1 + 2x_3 &= 2
\end{align*}
\]

Exercise T5: Prove by induction on \(n \in \mathbb{N}\) that
\[
\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}, \quad (b) \sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)} = \frac{n}{3n+1}.
\]

Exercise T6: Let \(n\) be a natural number. Proof the following statements by mathematical induction:
(a) \(\sum_{k=2}^{n} \frac{4}{(k-1)k(k+1)} = 1 - \frac{2}{n(n+1)}, \quad n \geq 2\),
(b) \(\sum_{l=0}^{n} \binom{n}{l} = 2^n, \quad n \geq 0\).

For detailed information regarding this course please check the page
http://www.math.kit.edu/iag6/lehre/am12012w/en

Tutorial date: Wednesday, October 31, 2012, 14:00-15:30 pm.