Worksheet No.7
Advanced Mathematics I

Exercise 31: Analyze the following series for convergence:

(a) \( \sum_{n=0}^{\infty} \left( \frac{3 + 4i}{6} \right)^n \),
(b) \( \sum_{n=1}^{\infty} \frac{n^2(n+1)^2}{n!} \),
(c) \( \sum_{n=8}^{\infty} \frac{n + 7\sqrt{n}}{n^3 - n} \).

Exercise 32: Calculate the respective first four partial sums \( s_1, s_2, s_3 \) and \( s_4 \) of the following series, and analyze these series for convergence:

(a) \( \sum_{\nu=1}^{\infty} \left( \frac{1}{\nu(\nu + 1)} - \frac{4}{\nu} \right) \),
(b) \( \sum_{\nu=1}^{\infty} (-1)^{\nu} \frac{2\nu + 1}{\nu(\nu + 1)} \).

Exercise 33: Show that the series

\( \sum_{k=0}^{\infty} (-1)^k \frac{k + 1}{2^k} \)

converges absolutely. Afterwards prove that the partial sums \( s_n \) of this series have the representation

\( s_n = \frac{1}{9} \left( 4 + (-1)^n \frac{3n + 5}{2^n} \right), \quad n \in \mathbb{N} \cup \{0\}, \)

using mathematical induction. Use this result for the calculation of the limit \( s \) of this series.

Exercise 34: Determine all \( \alpha \in \mathbb{R} \) for which the series

\( \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{\alpha - 1}{\alpha + 1} \right)^k \)

converges.

Hint: Consider the power series \( \sum_{k=1}^{\infty} \frac{1}{k} z^k \).

Exercise 35: A copy of the Tower of Babylon is made by piling the cubes \( W_n \) with edge length of \( \frac{1}{n} \) meter, where \( n \) runs through the natural numbers. In this process the base area of the \( n + 1 \)st cube is placed centered on the roof area of the \( n \)th cube.

(a) What will be the height of the tower?
(b) Can the tower be painted by a finite amount of dye?
(c) Can the builders manage with finite amount of concrete, assuming the cubes consist completely of concrete?

Due date: Please hand in your homework on Monday, December 10, 12:00, into the AM1-box in the student office in the International Department.
Exercise T20: Analyze the series for convergence using the ratio test in (a) and using the root test in (b) and (c):

(a) \( \sum_{k=0}^{\infty} \frac{2^k \cdot 3^k}{k! \cdot (2k+1)} \),  
(b) \( \sum_{k=0}^{\infty} \left( \frac{9}{10} + \frac{1}{k} \right)^k \),  
(c) \( \sum_{k=0}^{\infty} \frac{4^k (1 + 2k)^k (1 - k)^k}{(3 + k)^{2k} 2^k} \).

Exercise T21: Consider the series \( \sum_{n=0}^{\infty} a_n \), where

\[
a_n = \begin{cases} 
-\frac{1}{2n}, & n \text{ even} \\
\frac{1}{4^n}, & n \text{ odd}
\end{cases}
\]

(a) Show that the series converges absolutely using the comparison test.
(b) Verify that the root test shows absolute convergence of this series, too, while the ratio test is inconclusive.
(c) Determine the precise value of the series choosing a suitable decomposition.

Exercise T22: Show that the series \( \sum_{n=1}^{\infty} \frac{(-n)^n}{(n + 1)^{n+1}} \) converges using the Leibniz’ test.

Hint: Use the monotonicity of the sequence \( (1 + \frac{1}{n})^n \).

For detailed information regarding this course please check the page

http://www.math.kit.edu/iag6/lehre/am12012w/en

Tutorial date: Wednesday, December 5, 2012, 14:00-15:30 pm.