Exercise 36: Consider the polynomial \( f(x) = x^4 + 5x^3 - 8x^2 + 1 - (x - 2)^3 \).

(a) Determine the expansion of \( f \) about \( x_1 = 1 \) as well as the expansion about \( x_2 = -1 \).

(b) Represent \( f \) as a product of linear polynomials. What do you infer from that about the behaviour of \( f \) in the intervals \([1, \infty)\) and \((-\infty, -3]\)?

Exercise 37: Determine the radius and the interval of convergence of the following power series:

(a) \( \sum_{k=0}^{\infty} \frac{k + 2}{2k} x^k \),  
(b) \( \sum_{k=1}^{\infty} \frac{(2 + x)^{2k}}{(2 + \frac{1}{x})^k} \),  
(c) \( \sum_{k=0}^{\infty} \frac{3k+2}{2k} x^k \).

Exercise 38: The functions \( \cosh \), \( \sinh \) satisfy

\[
\cosh(x) = \frac{1}{2}(e^x + e^{-x}), \quad \sinh(x) = \frac{1}{2}(e^x - e^{-x}), \quad x \in \mathbb{R}, \quad \text{with} \quad e^x := \exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.
\]

(a) Prove the identity \( \cosh(2x) = [\cosh(x)]^2 + [\sinh(x)]^2 \) using the Cauchy product formula.

(b) Is there a shorter way to prove the identity?

(c) Determine the power series for \( f(x) = [\cosh(x)]^2 - [\sinh(x)]^2 \) expanded about \( x_0 = 0 \).

Exercise 39: Let the power series expansion of \( f(x) = e^{(x-x_0)} \) \( \frac{1}{1-(x-x_0)} \), \( x \in \mathbb{R} \setminus \{1+x_0\} \), about \( x_0 \in \mathbb{R} \) be \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \).

(a) Show that \( a_n = \sum_{k=0}^{n} \frac{1}{k!} \).

(b) For which \( x \in \mathbb{R} \) does the power series converge?

Exercise 40: Santa Claus, suffering from rheumatism, is fed up with all the snow. Since an early retirement is out of question, he moves from the north pole to the Maledive Islands and decides to deliver his presents henceforth only to places where the total quantity of snow in the coming years is bounded. His personal weather forecast predicts a snow quantity in the year \( 2000 + n \) proportional to

\[
a_n(x) := \frac{2^n x^{2n}}{\left(1 + \frac{x}{2}\right)^n},
\]

where \( x \in \mathbb{R} \) is the distance in the north-south direction along the surface of the earth (in 10,000 km) from the equator to a given location.

(a) Compute the radius of convergence of the power series \( \sum_{n=0}^{\infty} a_n(x) \).

(b) Will Santa Claus continue to deliver presents to Karlsruhe?

Hint: The latitude of Karlsruhe is 49°, the circumference of the earth is 40,000 km.

Due date: Please hand in your homework on Monday, December 17, 12:00, into the AM1-box in the student office in the International Department.
Exercise T23: Consider the function

\[ f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{1}{8} x^3 + \frac{3}{8} x^2 - \frac{9}{8} x + \frac{5}{8}. \]

Determine the expansion of \( f \) about \( x_1 = 1 \) as well as the expansion about \( x_2 = 3 \). What do you infer from that about the behaviour of \( f \) in the intervals \([-5, \infty)\) and \((\infty, 3]\)?

Exercise T24: Determine for \( z \in \mathbb{C} \) the radius and the disk of convergence of the power series:

(a) \( \left( \sum_{k=0}^{\infty} \frac{z^k}{3(k+2)!} \right) \),  
(b) \( \left( \sum_{k=1}^{\infty} \frac{z^{2k} \cdot 2^k}{(1 + \frac{1}{z})^k} \right) \),  
(c) \( \left( \sum_{k=0}^{\infty} k^k z^k \right) \).

Exercise T25:

(a) Determine the power series of the rational function \( f : \mathbb{C} \setminus \{1\} \to \mathbb{C} \), where \( f(z) = \frac{1+z^2}{1-z} \) is expanded about \( z_0 = 0 \).

(b) Compute the radius of convergence of the series.

(c) For which \( z \in \mathbb{C} \) does the power series converge?

For detailed information regarding this course please check the page


**Tutorial date:** Wednesday, December 12, 2012, 14:00-15:30 pm.