Exercise Sheet No. 8
Advanced Mathematics I

Exercise 36: For each of the following power series find all $z \in \mathbb{C}$ where the series converges

(a) $\sum_{n=0}^{\infty} \frac{z^n}{2^n (n!)^2}$,  
(b) $\sum_{n=1}^{\infty} \frac{(z - 1)^{2n}}{(1 + \frac{k}{n})^n}$,  
(c) $\sum_{n=1}^{\infty} n^n (z + 2)^n$.

Exercise 37: Expand the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$, $f(x) = \frac{e^x}{1 - x}$ as a power series $\sum_{n=0}^{\infty} a_n x^n$ centered at 0.

(a) Show that $a_n = \sum_{k=0}^{n} \frac{1}{k!}$ for all $n \in \mathbb{N}_0$.

(b) Find all $x \in \mathbb{R}$ where the power series converges.

Exercise 38: Consider the functions $\frac{1}{z + 1}$, $z \in \mathbb{C} \setminus \{-1\}$, and $\frac{1}{z + 2}$, $z \in \mathbb{C} \setminus \{-2\}$.

(a) Expand both functions as power series centered at $z_0 = 1$ and find the radii of convergence.

(b) Expand the function $\left(\frac{1}{z + 1}\right) \cdot \left(\frac{1}{z + 2}\right)$ as a power series centered at $z_0 = 1$ using the Cauchy product. Find a closed form representation of the coefficients!

*Hint:* You may use the identity from Exercise T24 without proof.

(c) Expand the function $\left(\frac{1}{z + 1}\right) \cdot \left(\frac{1}{z + 2}\right)$ as a power series centered at $z_0 = 1$ using partial fraction decomposition.

*Hint:* This means find numbers $a, b$ such that $\left(\frac{1}{z + 1}\right) \cdot \left(\frac{1}{z + 2}\right) = \left(\frac{a}{z + 1}\right) + \left(\frac{b}{z + 2}\right)$ for all $z$ and proceed.

(d) Find all $z \in \mathbb{C}$ where the Cauchy product from b) converges.

Exercise 39: Prove the following identities for all $z \in \mathbb{C}$

(a) $\cos(iz) = \cosh z$ and $\cos z = \cosh iz$,

(b) $\cos \bar{z} = \overline{\cos z}$,

(c) $\sinh(2z) = 2 \sinh z \cosh z$.

Exercise 40: Consider the real sequence $(a_n)$ given by $a_0 \geq -4$ and $a_{n+1} = 2 + \sqrt{4 + a_n}$ for all $n \in \mathbb{N} \cup \{0\}$.

(a) Prove the following statements.

*•* If $-4 \leq a_0 \leq 5$, then $a_n \in [2, 5]$ for all $n \in \mathbb{N}$.

*•* If $a_0 \geq 5$, then $a_n \geq 5$ for all $n \in \mathbb{N}$.

(b) Why does the sequence converge for all initial values $a_0 \geq -4$?

(c) Find the limit of $(a_n)$ for each initial value $a_0 \geq -4$.

**Due date:** Your written solutions are due by 12:30 P.M. on Monday, 12th January 2015. Please put them into the box in the student office.
Exercise T22: Write each of the following complex numbers \( z \) in the form \( z = re^{i\phi} \) with \( r, \phi \in \mathbb{R} \), \( r \geq 0 \) and \( \phi \in (-\pi, \pi] \)

(a) \(-i\),
(b) \(-1 + i\),
(c) \((1 - i)e^{-i\pi}\),
(d) \(\frac{3 + 4i}{1 - i}\),
(e) \(\frac{1 - i}{i + 2}\).

*Hint:* \(\arctan(-7) \approx -1.43\) and \(\arctan(-3) \approx -1.25\)

Exercise T23: Consider the function \( f: \mathbb{R} \rightarrow \mathbb{R} \) defined by

\[
 f(x) = \|x - 2| - 1\| + 2.
\]

Find a subdivision of \( \mathbb{R} \) into the smallest number of intervals such that the restriction of \( f \) to each of these intervals is invertible. Give a closed form representation of each of the inverses and sketch them.

Exercise T24: For each of the following functions find all points \( x \in \mathbb{R} \) where the function is continuous

a) \( f(x) = \begin{cases} \frac{|x-2|}{x(x-2)}, & x \in \mathbb{R} \setminus \{0,2\} \\ 1/2, & x = 2 \end{cases} \),

b) \( g(x) = \begin{cases} 2(x + 1)^2, & x < -1 \\ -x, & x \in [-1,1] \\ x^2 - 2x, & x > 1 \end{cases} \).

Exercise T25: Consider \( x, y \in \mathbb{R} \setminus \{0\} \) with \( x \neq y \). Prove the following identity by induction for all \( n \in \mathbb{N} \)

\[
\sum_{k=0}^{n} \left( \frac{1}{x} \right)^k \left( \frac{1}{y} \right)^{n-k} = \frac{xy^{-k} - yx^{-k}}{x - y}.
\]

For detailed information regarding this course visit the following web page:

www.math.kit.edu/iag6/lehre/am12014w/en