Exercise 41: Use appropriate power series centered at 0 and prove convergence of these series to calculate the following limits

\[
\begin{align*}
(a) \quad \lim_{x \to 0} \frac{1 - \cos(x)}{x \sin(x)}, \quad & \quad (b) \quad \lim_{x \to 0} \frac{\cos(x) - 1 + \frac{x^2}{2}}{\exp(x^4) - 1}.
\end{align*}
\]

Hint: Theorem 4.19, (c) states that power series define continuous functions in the disc of convergence.

Exercise 42:

(a) Use the power series representation of \(\cosh\) and \(\sinh\) about 0 and the Cauchy Product to prove the following identity for all \(z \in \mathbb{C}\)

\[
\sinh(2z) = 2 \cosh(z) \sinh(z).
\]

Hint: You may use the identity \(\sum_{k=0}^{m} (\frac{2m+1}{2k+1}) = 2^{2m}\) without proof.

(b) Prove the following special case of Theorem 4.24 (f) for all \(z \in \mathbb{C}\)

\[
\cosh(2z) = (\cosh z)^2 + (\sinh z)^2.
\]

Exercise 43: Find real- and imaginary part of each solution \(z \in \mathbb{C}\) of the following equation

\[
e^{iz} - 1 = 1 - 2e^{-iz}.
\]

For which \(z \in \mathbb{C}\) is the term on the left hand side not defined?

Exercise 44: Find all solutions \(z \in \mathbb{C}\) of the following equation

\[
3^{-2z} + 1 = 2 \cosh(z \ln 3).
\]

Exercise 45: Consider the following power series

\[
\sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1}.
\]

(a) Find all \(x \in \mathbb{R}\) such that the series converges.

(b) Find an index \(N\) such that for \(n \geq N\) the \(n\)'th partial sum differs from \(\ln(5/4)\) by at most \(\frac{1}{1000}\).

Hint: You may use the fact that the given series is a power series expansion of \(\ln(1 + x)\) without proof.

Due date: Your written solutions are due by 12:30 P.M. on Monday, 19th January 2015.
Please put them into the box in the student office.
Exercise T25: Use a power series representation of \( \sin(x) \) to determine the following limit

\[
\lim_{{x \to 0}} \frac{x^3}{\sin(x) - x}.
\]

Hint: Theorem 4.19, (c) states that power series define continuous functions in the disc of convergence.

Exercise T26: Find all solutions \( z \in \mathbb{C} \) of the following equations

\begin{align*}
(a) \quad & \cosh(z) = -1, \\
(b) \quad & \cos(z) + \sin(z) = \frac{3}{2}.
\end{align*}

Exercise T27: Prove the following identities for all \( z \in \mathbb{C} \) where \( z = x + iy \)

\[
\sin(z) = \sin(x) \cosh(y) + i \cos(x) \sinh(y), \quad \cos(z) = \cos(x) \cosh(y) - i \sin(x) \sinh(y).
\]

For detailed information regarding this course visit the following web page:

www.math.kit.edu/iag6/lehre/am12014w/en