Karlsruhe Institute of Technology
Institute for Algebra und Geometry
Prof. Axenovich Ph.D.

1	2	3	4	5	Σ

Karlsruhe, October 18, 2018

Name: ....

Exercise Sheet No. 1
Advanced Mathematics I

Exercise 1:

Consider the following sets of real numbers: $A = [-4, -1]$, $B = \{-3, -2, 0, 3\}$ und $C = [-2, 4]$.

(a) Determine the sets $A \cup C$, $A \cap B$ and $(A \backslash C) \cup (C \backslash A)$,

(b) Find a maximal set M satisfying $(M \backslash A) \subseteq (B \cup C)$.

Solution 1: (a) From the definitions we have the following:

$$\begin{aligned} A \cup C &= [-4, -1] \cup [-2, 4] = [-4, 4]\,, \\ A \cap B &= [-4, -1] \cap \{-3, -2, 0, 3\} = \{-3, -2\}\,, \\ (A \backslash C) \cup (C \backslash A) &= [-4, -2) \cup (-1, 4]\,. \end{aligned}$$

(b) First, we make some simple observations. Any maximal set M must contain A, since otherwise adding a missing element from A to M leaves $M \setminus A$ unchanged. Second, any maximal set M must contain all of $B \cup C$. Indeed, otherwise if we add any missing element from $B \cup C$ to M we still have the required containment. Thus, every maximal set M satisfies $A \cup B \cup C \subset M$.

We now show $A \cup B \cup C$ is maximal. (The first paragraph is not required to solve this problem, but it provides intuition and shows that the maximal set M we find is unique.) Suppose there was an element $x \notin A \cup B \cup C$ such that $(M \cup \{x\}) \setminus A \subseteq B \cup C$. Since $x \notin A$ we have that $x \in (M \cup \{x\}) \setminus A$. Then by the assumed containment, we must have $x \in B \cup C$. However, this contradicts that $x \notin B \cup C$ (as $x \notin A \cup B \cup C$). It follows that $A \cup B \cup C$ is maximal.

Exercise 2: Solve the following inequality and equality for x:

(a) $(x - 5)^3(x + 1) \geq 0$,

(b) $|x| = x^3 + 2x^2 - 3x$.

Solution 2:

(a) $(x - 5)^3(x + 1)$ is nonnegative if and only if $(x - 5)^3 \geq 0$ and $x + 1 \geq 0$, or $(x - 5)^3 \leq 0$ and $x + 1 \leq 0$. In the first case, we have $\mathcal{L}_1 = [5, \infty) \cap [-1, \infty) = [5, \infty)$. Analogously, in the second we havel $\mathcal{L}_2 = (-\infty, 5] \cap (-\infty, -1] = (-\infty, -1]$. Combining the results gives

$$\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 = (-\infty, -1] \cup [5, \infty)\,.$$

(b) We condition on the following two possibilities: $x \geq 0$ und $x < 0$. In the first case we have $|x| = x$ and so:

$$x = x^3 + 2x^2 - 3x \text{ and } x \geq 0.$$

This is true if and only if $0 = x^3 + 2x^2 - 4x$ and $x \geq 0$, equivalently $0 = x((x + 1)^2 - 5)$ and $x \geq 0$. This yields

$$\mathcal{L}_+ = \left\{-1 - \sqrt{5}, 0, -1 + \sqrt{5}\right\} \cap [0, \infty) = \left\{0, -1 + \sqrt{5}\right\}.$$

For the case $x < 0$ we have $|x| = -x$, and so $-x = x^3 + 2x^2 - 3x$ and $x < 0$, or equivalently

$$0 = x^3 + 2x^2 - 2x \text{ and } x < 0.$$

This is true if and only if $0 = x((x + 1)^2 - 3)$ and $x < 0$. It follows that

$$\mathcal{L}_- = \left\{-1 - \sqrt{3}, 0, -1 + \sqrt{3}\right\} \cap (-\infty, 0) = \left\{-1 - \sqrt{3}\right\}.$$

Thus, the final solution is

$$\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_- = \{-1 - \sqrt{3}, 0, -1 + \sqrt{5}\}.$$

Exercise 3: Write each of the following sets as a union of intervals:

$$U = \left\{x \in \mathbb{R} : \left((x-4)^2 - 10\right)^2 \geq 36\right\}, V = \{x \in \mathbb{R} : |2x+6| + |2x-6| - |x+1| - |x-1| > 8\}.$$

Solution 3: Set U: We solve the inequality in several steps as follows.

$$\left((x-4)^2 - 10\right)^2 \geq 36 \text{ which occurs when } (x-4)^2 - 10 \geq 6 \text{ or } (x-4)^2 - 10 \leq -6.$$

We distinguish two cases: $(x-4)^2 - 10 \geq 6$ or $(x-4)^2 - 10 \leq -6$.

In the first case:

$$(x-4)^2 - 10 \geq 6, \text{ so } (x-4)^2 \geq 16, \text{ thus } x-4 \geq 4 \text{ or } x-4 \leq -4, \text{ finally } \underline{x \geq 8 \text{ or } x \leq 0}.$$

In the second case:

$$(x-4)^2 - 10 \leq -6, \text{ so } (x-4)^2 \leq 4, \text{ thus } |x-4| \leq 2, \text{ so } -2 \leq x-4 \leq 2, \text{ finally } \underline{2 \leq x \leq 6}.$$

Combining we obtain

$$U = \{x \in \mathbb{R}\colon x \leq 0 \text{ or } 2 \leq x \leq 6 \text{ or } x \geq 8\} = (-\infty, 0] \cup [2,6] \cup [8, \infty).$$

Set V: The points where one of the arguments of the absolute value functions change sign are $x = -3, -1, 1$ are 3. Thus, we will split into cases based on the value of x relative to these values.

$$\begin{aligned}
&|2x+6| + |2x-6| - |x+1| - |x-1| \\
&= \begin{cases} -(2x+6) + (-(2x-6)) - (-(x+1)) - (-(x-1)), & \text{for } x \leq -3, \\ (2x+6) + (-(2x-6)) - (-(x+1)) - (-(x-1)), & \text{for } -3 < x \leq -1, \\ (2x+6) + (-(2x-6)) - (x+1) - (-(x-1)), & \text{for } -1 < x \leq 1, \\ (2x+6) + (-(2x-6)) - (x+1) - (x-1), & \text{for } 1 < x \leq 3 \\ (2x+6) + (2x-6) - (x+1) - (x-1), & \text{for } 3 < x, \end{cases} \\
&= \begin{cases} -2x, & \text{for } x \leq -3, \\ 2x+12, & \text{for } -3 < x \leq -1, \\ 10, & \text{for } -1 < x \leq 1, \\ -2x+12, & \text{for } 1 < x \leq 3 \\ 2x, & \text{for } 3 < x. \end{cases}
\end{aligned}$$

$$\text{So } V = \left\{ x \in \mathbb{R}\colon \begin{array}{rlll} -2x > 8 & \text{and} & x \leq -3, & \text{or} \\ 2x+12 > 8 & \text{and} & -3 < x \leq -1, & \text{or} \\ 10 > 8 & \text{and} & -1 < x \leq 1, & \text{or} \\ -2x+12 > 8 & \text{and} & 1 < x \leq 3 & \text{or} \\ 2x > 8 & \text{and} & 3 < x. & \end{array} \right\} = \left\{ x \in \mathbb{R}\colon \begin{array}{rl} x < -4 & \text{or} \\ -2 < x \leq -1 & \text{or} \\ -1 < x \leq 1 & \text{or} \\ 1 < x < 2 & \text{or} \\ 4 < x. & \end{array} \right\}.$$

Combining we obtain

$$V = \{x \in \mathbb{R}\colon x < -4 \text{ or } -2 < x < 2 \text{ or } x > 4\} = (-\infty, -4) \cup (-2, 2) \cup (4, \infty).$$

Set $U \cap V$: We have

$$U \cap V = \big((-\infty, 0] \cup [2,6] \cup [8,\infty)\big) \cap \big((-\infty,-4) \cup (-2,2) \cup (4,\infty)\big) = (-\infty,-4) \cup (-2,0] \cup (4,6] \cup [8,\infty).$$

Exercise 4: Evaluate the following sums:

$$\text{(a)} \quad \sum_{n=7}^{42} \left(\frac{1}{3}\right)^n, \qquad \text{(b)} \quad \sum_{m=-1}^{8} (n+1)^3 \text{ for } n \in \mathbb{N}, \qquad \text{(c)} \quad \sum_{\mu=0}^{1} \sum_{\nu=2}^{4} \frac{1}{\mu + \nu^2}.$$

Solution 4:

$$\text{(a)} \sum_{n=7}^{42} \left(\frac{1}{3}\right)^n \overset{\text{Index change}}{=} \sum_{n=0}^{35} \left(\frac{1}{3}\right)^{n+7} = \left(\frac{1}{3}\right)^7 \sum_{n=0}^{35} \left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^7 \frac{1 - \left(\frac{1}{3}\right)^{36}}{1 - \frac{1}{3}} = \left(\frac{1}{3}\right)^7 \frac{3 - \left(\frac{1}{3}\right)^{35}}{2} = \frac{3^{36} - 1}{2 \cdot 3^{42}}.$$

$$\text{(b)} \sum_{m=-1}^{8} (n+1)^3 = (n+1)^3 \cdot \sum_{m=-1}^{8} 1 = 10\,(n+1)^3, \quad \text{noting that } n \text{ is fixed and not the index variable.}$$

(c) $\displaystyle\sum_{\mu=0}^{1}\left(\sum_{\nu=2}^{4}\frac{1}{\mu+\nu^2}\right)=\left(\frac{1}{0+2^2}+\frac{1}{3^2}+\frac{1}{4^2}\right)+\left(\frac{1}{1+4}+\frac{1}{1+9}+\frac{1}{1+16}\right)=\frac{9577}{12240}.$

Exercise 5:

(a) Change the indices of the summations in the following expression so that the expression may be written with only one summation sign.

$$\sum_{k=2}^{23}(k-1)^2+\sum_{\ell=-2}^{19}2(\ell+3)+\sum_{m=10}^{31}1.$$

Then evaluate the sum with the help of some results from the lecture.

(b) Prove the following identity for all a, $b\in\mathbb{R}$ and $n\in\mathbb{N}$ by means of changing indices:

$$(a-b)\sum_{k=0}^{n}a^kb^{n-k}=a^{n+1}-b^{n+1}.$$

Solution 5: a) We apply the index transformations $n=k-1$, $n=\ell+3$ and $n=m-9$. Then we can write the expression as one sum.

$$\sum_{k=2}^{23}(k-1)^2+\sum_{\ell=-2}^{19}2(\ell+3)+\sum_{m=10}^{31}1=\sum_{n=1}^{22}n^2+\sum_{n=1}^{22}2n+\sum_{n=1}^{22}1=\sum_{n=1}^{22}(n^2+2n+1).$$

Applying the binomial theorem and another index change we have

$$\sum_{n=1}^{22}(n^2+2n+1)=\sum_{n=1}^{22}(n+1)^2=\sum_{n=2}^{23}n^2=\left(\sum_{n=1}^{23}n^2\right)-1\overset{\text{Vorl.}}{=}\frac{1}{6}\cdot23\cdot24\cdot47-1=4323.$$

b) We multiply, then we pull out the last term of the first sum and the first term of the second sum.

$$(a-b)\sum_{k=0}^{n}a^kb^{n-k}=\sum_{k=0}^{n}a^{k+1}b^{n-k}-\sum_{k=0}^{n}a^kb^{n-k+1}=a^{n+1}+\sum_{k=0}^{n-1}a^{k+1}b^{n-k}-\sum_{k=1}^{n}a^kb^{n-k+1}-b^{n+1}.$$

Now we use a change of index $k'=k-1$, that is $k=k'+1$:

$$\sum_{k=1}^{n}a^kb^{n-k+1}=\sum_{k'=0}^{n-1}a^{k'+1}b^{n-k'}.$$

Plugging this in above we see that the sums cancel out, and we obtain $(a-b)\sum_{k=0}^{n}a^kb^{n-k}=a^{n+1}-b^{n+1}$, as required.

Due date: Your written solutions are due at 14:00 on Tuesday, **October 30, 2018**. Please submit them in the beginning of the problem class.

Problem Session: 14:00 Tuesday, October 23, 2018

Website: For detailed information regarding this course visit the following web page:

`http://www.math.kit.edu/iag6/edu/am12018w/en`