Karlsruhe Institute of Technology
Institute for Algebra und Geometry
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Karlsruhe, October 26, 2018

Name: ....

Exercise Sheet No. 2
Advanced Mathematics I

Exercise 6:

(a) Let $n, m, r \in \mathbb{N}$ and $n \geq m \geq r \geq 0$. Prove the following equality about binomial coefficients.

$$\binom{n}{m} \cdot \binom{m}{r} = \binom{n}{r} \cdot \binom{n-r}{m-r}.$$

(b) Evaluate the following summations:

(i)

$$\sum_{k=0}^{5} \binom{5}{k},$$

(ii)

$$\sum_{n=3} 5 \frac{\binom{n}{3}}{n!}.$$

Solution 6:

(a) For $n \geq m \geq r \geq 0$ we have the following:

$$\begin{aligned}\binom{n}{m} \cdot \binom{m}{r} &= \frac{n!}{m!\,(n-m)!} \cdot \frac{m!}{r!\,(m-r)!} = \frac{n!}{r!\,(n-m)!\,(m-r)!} \\ &= \frac{n!}{r!\,(n-r)!} \cdot \frac{(n-r)!}{([n-r]-[m-r])!\,(m-r)!} = \binom{n}{r} \cdot \binom{n-r}{m-r}.\end{aligned}$$

(b) (i)We make use of the identity: $\binom{n}{k} = \binom{n}{n-k}$.

$$\begin{aligned}\sum_{k=0}^{5} \binom{5}{k} &= \left(\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}\right) = 2 \cdot \left(\binom{5}{0} + \binom{5}{1} + \binom{5}{2}\right) \\ &= 2 \cdot \left(\frac{5!}{5!0!} + \frac{5!}{5!4!} + \frac{5!}{5!3!}\right) = 2 \cdot (1 + 5 + 10) = 32.\end{aligned}$$

(ii)

$$\sum_{n=3}^{5} \frac{\binom{n}{3}}{n!} = \sum_{n=3}^{5} \frac{n!}{3!(n-3)!n!} = \sum_{n=3}^{5} \frac{1}{3!(n-3)!} = \frac{1}{3!0!} + \frac{1}{3!1!} + \frac{1}{3!2!} = \frac{1}{6} + \frac{1}{6} + \frac{1}{12} = \frac{5}{12}$$

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Exercise 7:

(a) Prove that the sum of three consecutive natural numbers is always divisible by 3.

(b) Prove by contradiction that for each prime p, $\sqrt{p}$ is irrational (recall that 1 is not prime).

Solution 7:

(a) Let $r, r+1, r+2$ be 3 consecutive natural numbers. Their sum is $r + (r+1) + (r+2) = 3r + 3 = 3(r+1)$ which is 3 times an integer. Thus, 3 divides the sum of 3 consecutive natural numbers.

(b) Let p be a prime. Suppose by contradiction that $\sqrt{p} = a/b$ where a and b are natural numbers and a/b is reduced. Let $a = \prod_i p_i$, $b = \prod_i q_i$ be representations of a and b as products of primes as guaranteed to exist by the fundamental theorem of arithmetic. Rewrite $\sqrt{p} = a/b$ as $\sqrt{p} \prod_i q_i = \prod_i p_i$. Squaring both sides we obtain $p \prod_i q_i^2 = \prod_i p_i^2$ a contradiction since p appears an odd number of times on the left and an even number of times on the right.

Exercise 8: Prove the following equality by induction:

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Solution 8 *Base Case:* $\sum_{k=1}^{1} k^3 = 1 = 1^3$.

Induction Step: Suppose we have the result for $n \in \mathbb{N}$, we will prove it also holds for $n+1$.

Considering the first n terms and the final term of the sum separately, we have by the inductive hypothesis:

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 = \frac{(n+1)^2}{4}\left(n^2+4n+4\right) = \frac{(n+1)^2\,(n+2)^2}{4},$$

which is the required form.

Exercise 9: Prove the following identities by induction:

(a)

$$\sum_{k=0}^{n}(2k+1) = (n+1)^2,$$

(b)

$$\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}.$$

Solution: a) *Base Case:* $n = 0 : \sum_{k=0}^{0} 2k+1 = 1 = (0+1)^2$.

Induction Step: Assume that for $n \in \mathbb{N}$ we have $\sum_{k=0}^{n}(2k+1) = (n+1)^2$. We will prove $\sum_{n=0}^{n+1}(2k+1) = (n+2)^2$ holds. Using $(a+b)^2 = a^2 + 2ab + b^2$, we have:

$$\begin{aligned}\sum_{k=0}^{n+1}(2n+1) &= \sum_{k=0}^{n}(2k+1) + [2(n+1)+1] \overset{\text{I.V.}}{=} (n+1)^2 + 2(n+1) + 1\\ &= (n+1+1)^2 = (n+2)^2.\end{aligned}$$

b) *Base Case:* $n = 1 : \sum_{n=1}^{1} \frac{1}{1\cdot 3} = \frac{1}{3} = \frac{1}{2\cdot 1+1}$.

Induction Step:

$$\begin{aligned}\sum_{k=1}^{n+1}\frac{1}{(2k-1)(2k+1)} &= \sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)} + \frac{1}{(2n+1)(2n+3)} \overset{\text{I.V.}}{=} \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)}\\ &= \frac{n(2n+3)+1}{(2n+1)(2n+3)} = \frac{2n^2+3n+1}{(2n+1)(2n+3)} = \frac{(2n+1)(n+1)}{(2n+1)(2n+3)} = \frac{n+1}{2(n+1)+1}.\end{aligned}$$

Exercise 10: Prove the following identity for $n \in \mathbb{N}$:

$$\sum_{k=1}^{n}\frac{(2k)! - 2\cdot(2k-2)!}{2^k} = \frac{(2n)!}{2^n} - 1.$$

Solution: First we consider the base case: $n = 1$.

$$\sum_{k=1}^{1}\frac{(2k)! - 2\,(2k-2)!}{2^k} = \frac{2-2}{2} = 0 = \frac{2!}{2} - 1.$$

Now, suppose the result holds for $n \in \mathbb{N}$, we will show it holds for $n+1$.

$$\begin{aligned}\sum_{k=1}^{n+1}\frac{(2k)! - 2(2k-2)!}{2^k} &= \sum_{k=1}^{n}\frac{(2k)! - 2(2k-2)!}{2^k} + \frac{(2(n+1))! - 2(2(n+1)-2)!}{2^{n+1}}\\ &= \frac{(2n)!}{2^n} - 1 + \frac{(2(n+1))! - 2(2(n+1)-2)!}{2^{n+1}}\\ &= \frac{2(2n)! + (2(n+1))! - 2(2(n+1)-2)!}{2^{n+1}} - 1\\ &= \frac{(2(n+1))!}{2^{n+1}} - 1.\end{aligned}$$

Due date: Your written solutions are due at 14:00 on Tuesday, **6 November, 2018**.
Please submit them at the beginning of the problem session
or in the box in J101 (note the box will be emptied before the problem session).

Problem Session: 14:00 Tuesday, November 6, 2018

Website: For detailed information regarding this course visit the following web page:

`http://www.math.kit.edu/iag6/edu/am12018w/en`