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Karlsruhe, November 23, 2018

Name: ....

Exercise Sheet No. 6
Advanced Mathematics I

Exercise 26: Consider the polynomial $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := \frac{1}{8}\,x^3 + \frac{3}{8}\,x^2 - \frac{9}{8}\,x + \frac{5}{8}$.

(a) Express $f(x)$ in terms of $x-1$ and $x+3$. Use this representation to discuss the behavior of f on the intervals $[-5, \infty)$ and $(-\infty, 3]$. Do not use the derivative f' for this discussion.

(b) Use a sketch of f to find intervals on which f has an inverse function. Also sketch the inverse on these intervals.

Solution:

(a) First we will expand the function about the point $x_1 = 1$. We look for an expression in the form:

$$f(x) = \sum_{k=0}^{3} a_k (x-1)^k \qquad \text{with } a_k \in \mathbb{R} \text{ for } k = 0, .., 3$$

We replace every x with the equivalent form $x + 1 - 1$ obtaining:

$$f(x) = \frac{1}{8}(x-1+1)^3 + \frac{3}{8}(x-1+1)^2 - \frac{9}{8}(x-1+1) + \frac{5}{8}$$

Now we expand using the binomial formula

$$f(x) = \frac{1}{8}\big[(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1\big] + \frac{3}{8}\big[(x-1)^2 + 2(x-1) + 1\big] - \frac{9}{8}\big[(x-1) + 1\big] + \frac{5}{8}$$

We obtain

$$f(x) = \frac{1}{8}\left[(x-1)^3 + 6(x-1)^2\right].$$

Now we expand the function about the point $x_2 = -3$:

$$\begin{aligned} f(x) &= \frac{1}{8}(x+3-3)^3 + \frac{3}{8}(x+3-3)^2 - \frac{9}{8}(x+3-3) + \frac{5}{8} \\ &= \frac{1}{8}\big[(x+3)^3 + 3(x+3)^2(-3) + 3(x+3)(-3)^2 + (-3)^3\big] \\ &\quad + \frac{3}{8}\big[(x+3)^2 + 2(x+3)(-3) + (-3)^2\big] - \frac{9}{8}\big[(x+3) - 3\big] + \frac{5}{8} \\ &= \frac{1}{8}\big[(x+3)^3 - 6(x+3)^2 + 32\big] \end{aligned}$$

Looking at the expansion about $x = 1$, we see that at $x = 1$ there is a local minimum. Looking at the expansion about $x = -3$, we see that at $x = -3$ there is a local maximum. We deduce that f is increasing until to $x = -3$, then decreasing until $x = 1$, and then increasing.

(b) Since the function is monotone on the intervals $(-\infty, -3], [-3, 1]$ and $[1, \infty)$, these are the intervals on which f has an inverse.

Exercise 27: Decide whether the functions are injective, surjective or bijective. If the function is bijective, then find the inverse. Justify your answer.

(a) $f\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$, $f(x) = |x|$

(b) $g\colon \mathbb{N} \to \mathbb{N}$, $g(n) = n + 1$

(c) $h\colon \mathbb{N}_0 \to \mathbb{Z}$, $h(n) = \begin{cases} \frac{n}{2}, & \text{for } n \text{ even} \\ -\frac{n+1}{2}, & \text{for } n \text{ odd} \end{cases}$

Solution:

(a) f is surjective. Indeed, take any $y \in \mathbb{R}_{\geq 0}$, then $f(y) = |y| = y$. f is not injectice since $f(-1) = f(1) = 1$.

(b) The function g is injective. If we have $g(n) = g(m)$, then $n+1 = m+1$ so $n = m$. g is not surjective since 1 is not in the image.

(c) The function h is bijective. First we show h is injective. Assume $h(n) = h(m)$. If $h(n) = h(m) = 0$ then m and n are both 0. Assume $h(m)$ and $h(n)$ are both equal to a positive number k. Then m and n are even and $n/2 = k$ and $m/2 = k$. Thus, $n/2 = m/2$ and so $n = m$. Assume now that $h(m)$ and $h(n)$ are both equal to a negative number k. Then m and n are odd and $-\frac{n+1}{2} = k$ and $-\frac{m+1}{2} = k$, and we again have $m = n$.

Now we show h is surective. Let $k \in \mathbb{N}_0$. If $k \geq 0$, then $h(2k) = k$. If $k < 0$, then $h(-2k+1) = k$. Thus every element of $\mathbb{N}_0$ is in the image of h. Since h is injective and surjective it is bijective.

Exercise 28:

Using the sequence definition of continuity, find $w \in \mathbb{R}$ so that the function

$$f(x) = \frac{x\sqrt{x}-1}{\sqrt{x}-1}\,, \qquad x > 0,\ x \neq 1,$$

is continuous if we set $f(1) := w$.

Solution:

We have $f(x) = \frac{x\sqrt{x}-1}{\sqrt{x}-1} = \frac{(\sqrt{x})^3-1}{\sqrt{x}-1} = \frac{(\sqrt{x}-1)(x+\sqrt{x}+1)}{\sqrt{x}-1} = x + \sqrt{x} + 1$. Take a sequence (x_n) with $x_n > 0,\ x_n \neq 1$ and $x_n \to 1\,(n \to \infty)$, then we have $f(x_n) \to 1 + \sqrt{1} + 1 = 3\,(n \to \infty)$. Thus we must set $f(1) = w = 3$.

Now we show that f with $f(1)$ defined to be 3 is in fact continuous. Take a sequence (x_n) converging to 1, with $x_n > 0$, but now allowing $x_n = 1$ for some values of n. Let (x_{n_k}) be the subsequence where $x_{n_k} = 1$ and (x_{j_k}) be the subsequence where $x_{j_k} \neq 1$. Let $\epsilon > 0$, then since we know that as (x_{j_k}) converges to 1 we have $(f(x_{j_k}))$ converging to 3, we may find an $N \in \mathbb{N}$ such that $k \geq N$ implies $|f(x_{j_k}) - 3| < \epsilon$. Then for $k \geq N$ we have $|f(x_k) - 3| < \epsilon$ because if $x_k \neq 1$ it is a member of the subsequence (x_{j_k}) and otherwise it is equal to 1 (and so the difference $f(x_k) - 3$ is 0). Thus with the definition $f(1) = 3$ the function is continuous everywhere it is defined.

Exercise 29:

The function $f : \mathbb{R}\backslash\{-2\} \to \mathbb{R}$ is defined as

$$f(x) = \begin{cases} \dfrac{5x \cdot |x-3|}{x^2 - x - 6} & , \quad x \in \mathbb{R}\backslash\{-2, 1, 3\}\ , \\ y_1 & , \quad x = 1\ , \\ y_2 & , \quad x = 3\ . \end{cases}$$

Is it possible for f to be continuous at $x = 1$ and $x = 3$ with a suitable choice of y_1, y_2? Give the appropriate values, or show that none exist.

Solution:

Let (x_n) be a sequence converging to 1: $\lim\limits_{n\to\infty} x_n = 1$. Assume f is continuous, then we have

$$y_1 = f(1) = \lim_{n\to\infty} f(x_n)\,.$$

It follows that

$$y_1 = \lim_{n\to\infty} \frac{5x_n \cdot |x_n - 3|}{x_n^2 - x_n - 6} = \frac{5 \lim\limits_{n\to\infty} x_n \cdot \left|\lim\limits_{n\to\infty} x_n - 3\right|}{\left(\lim\limits_{n\to\infty} x_n\right)^2 - \lim\limits_{n\to\infty} x_n - 6} = \frac{5 \cdot 1 \cdot |1-3|}{1 - 1 - 6} = \frac{10}{-6} = -\frac{5}{3}.$$

Thus we must take $y_1 = -\frac{5}{3}$.

Now let (x_n) be a sequence converging to 3: $\lim\limits_{n\to\infty} x_n = 3$. We require

$$y_2 = f(3) = \lim_{n\to\infty} f(x_n)\,.$$

Consider the sequences

$$x_n^+ = 3 + \frac{1}{n} \text{ und } x_n^- = 3 - \frac{1}{n}.$$

We have

$$y_2 = \lim_{n\to\infty} f\left(x_n^+\right) = \lim_{n\to\infty} \left(5x_n^+\right) \cdot \lim_{n\to\infty} \frac{\frac{1}{n}}{9 + \frac{6}{n} + \frac{1}{n^2} - 3 - \frac{1}{n} - 6} = 15 \lim_{n\to\infty} \frac{1}{5 + \frac{1}{n}} = \frac{15}{5} = 3.$$

and aslo

$$y_2 = \lim_{n\to\infty} f\left(x_n^-\right) = \lim_{n\to\infty}\left(5x_n^-\right)\cdot \lim_{n\to\infty} \frac{\left|-\frac{1}{n}\right|}{9-\frac{6}{n}+\frac{1}{n^2}-3+\frac{1}{n}-6} = 15\lim_{n\to\infty}\frac{1}{-5+\frac{1}{n}} = \frac{15}{-5} = -3.$$

but this means that we cannot choose a value for $x = 3$ so that the function is continuous there.

Exercise 30:

Prove the following functions are Lipschitz-continuous and find the respective Lipschitz constant.

(a) $f(x) = \sqrt{1+4x},\ D = [0,4)$,

(b) $f(x) = x^2+4x-1,\ D = (-4,3)$,

(c) $f(x) = \sqrt{2x^2+1},\ D = [-2,1]$.

Solution:

(a) Let $x_1,\ x_2 \in [0,4)$. We have

$$|f(x_1)-f(x_2)| = |\sqrt{1+4x_1}-\sqrt{1+4x_2}| = \left|\frac{4\,(x_1-x_2)}{\sqrt{1+4x_1}+\sqrt{1+4x_2}}\right| \le \frac{4}{2\sqrt{2}}\,|x_1-x_2| = \sqrt{2}\,|x_1-x_2|\,.$$

The function is thus Lipschitz continuous and the constant $\sqrt{2}$ works.

(b) Let $x_1,\ x_2 \in (-4,3)$. We have

$$\begin{aligned}|f(x_1)-f(x_2)| &= \left|x_1^2+4x_1-1-\left(x_2^2+4x_2-1\right)\right| = |(x_1+x_2+4)|\ |x_1-x_2|\\ &\le (|x_1|+|x_2|+4)\ |x_1-x_2| \le (4+4+4)\ |x_1-x_2| = 12\,|x_1-x_2|\,.\end{aligned}$$

The function is thus Lipschitz continuous and the constant 12 works.

(c) For $x_1,\ x_2 \in [-2,1]$, we have

$$\begin{aligned}|f(x_1)-f(x_2)| &= |\sqrt{2x_1^2+1}-\sqrt{2x_2^2+1}| = \frac{\left|2(x_1^2-x_2^2)\right|}{\sqrt{2x_1^2+1}+\sqrt{2x_2^2+1}} = \frac{2\,|x_1+x_2|}{\sqrt{2x_1^2+1}+\sqrt{2x_2^2+1}}\,|x_1-x_2|\\ &\le \frac{2(|x_1|+|x_2|)}{1+1}\,|x_1-x_2| \le \frac{8}{2}\,|x_1-x_2| = 4\,|x_1-x_2|,\end{aligned}$$

The function is thus Lipschitz continuous and the constant 4 works.

Due date: Your written solutions are due at 14:00 on Tuesday, **4 December, 2018**. Please submit them at the beginning of the problem session.

Website: For detailed information regarding this course visit the following web page:

`http://www.math.kit.edu/iag6/edu/am12018w/en`