Karlsruhe Institute of Technology
Institute for Algebra und Geometry
Prof. Axenovich Ph.D.

16	17	18	19	20	$\sum$

Karlsruhe, November 8, 2018

Name: ....

Exercise Sheet No. 4
Advanced Mathematics I

Exercise 16: Let the sequence $(x_n)_n$ be recursively defined by

$$x_1 = 0, \quad x_{n+1} = \frac{1}{2}(1 - x_n^2), \quad n \in \mathbb{N}.$$

(a) Compute x_j explicitly for $j \in \{1, 2, 3, 4\}$ and show by mathematical induction that the following estimation is true for all $n \in \mathbb{N}$:

$$0 \leq x_n \leq \frac{1}{2}.$$

(b) Conclude that the inequality $|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|$ holds for all $n \in \mathbb{N}$, $n \geq 2$, and prove by mathematical induction that

$$|x_{n+1} - x_n| \leq \left(\frac{1}{2}\right)^n, \quad n \in \mathbb{N}.$$

Exercise 17: Let the sequences $(a_n)_n$ and $(b_n)_n$ be defined by:

$$a_n = n^2 + 1, \qquad b_n = \frac{n^3 + n^2 + 3n + 1}{n^4 + n^2 - 3}, \quad n \in \mathbb{N}.$$

Compute the first 6 terms of each sequence $(a_n)_n$ and $(b_n)_n$. Which of the sequences are bounded? Give a formal proof of your answer.

Exercise 18: Compute the limit of each of the following sequences

(a) $a_n = \frac{n^4 - 2}{n^2 + 4} + \frac{n^3(3 - n^2)}{n^3 + 1}$

(b) $b_n = \left(1 + \left(-\frac{3}{5}\right)^n\right) \cdot \left(\frac{10^n}{n!} - \frac{3n^2 + 1}{(2n+1)^2}\right)$

(c) $c_n = \sqrt[n]{17 \cdot 2^n}\left(\sqrt{n+1} - \sqrt{n}\right)$.

Exercise 19: Calculate the limits of the complex sequences

$$\text{(a) } a_n = 2 + \frac{3}{4i\,n} + \left(\frac{1}{2} + \frac{1}{3}i\right)^n, \quad n \in \mathbb{N}, \qquad \text{(b) } b_n = \frac{(3i\,n + 1)(2n + i)}{\sum\limits_{k=1}^{n} ik}, \quad n \in \mathbb{N}.$$

Exercise 20: Consider the sequence $(a_n)_n$ with $a_n = \frac{n-1}{n+1}$, $n \in \mathbb{N}$. Find an index N such that $|a_n - 1| < \varepsilon$ for every $n \geq N$, when

(a) $\varepsilon = \dfrac{1}{10}$ (b) $\varepsilon = \dfrac{1}{1000}$, (c) $\varepsilon > 0$ is arbitrary.

(d) Does the sequence $(a_n)_n$ converge? If so, what is the limit?

Due date: Your written solutions are due at 14:00 on Tuesday, **20 November, 2018**.
Please submit them at the beginning of the problem session
or in the box in J101 (note the box will be emptied before the problem session).

Website: For detailed information regarding this course visit the following web page:

`http://www.math.kit.edu/iag6/edu/am12018w/en`