Karlsruhe Institute of Technology
Institute for Algebra und Geometry
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Karlsruhe, December 20, 2018

Name: ....

Exercise Sheet No. 10
Advanced Mathematics I

Exercise 46: Prove the following identities using the addition theorems for sine and cosine:

(a) $\cot(a+b) = \frac{\cot a \cot b - 1}{\cot a + \cot b}$,

(b) $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$, $\quad x \in (-\pi, \pi)$.

Solution:

(a) Using the addition formulas for sine and cosine we have

$$\frac{\cot a \cot b - 1}{\cot a + \cot b} = \frac{\frac{\cos a \cos b}{\sin a \sin b} - 1}{\frac{\cos a}{\sin a} + \frac{\cos b}{\sin b}} = \frac{\frac{\cos a \cos b - \sin a \sin b}{\sin a \sin b}}{\frac{\cos a \sin b + \cos b \sin a}{\sin a \sin b}} = \frac{\cos(a+b)}{\sin(a+b)} = \cot(a+b).$$

(b) Using the addition formulas for sine and cosine and the identity $\sin^2(\frac{x}{2}) + \cos^2(\frac{x}{2}) = 1$ we have

$$\sin x = \sin(\frac{x}{2} + \frac{x}{2}) = 2\sin\frac{x}{2}\cos\frac{x}{2}$$
$$\cos x = \cos(\frac{x}{2} + \frac{x}{2}) = \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2}) - \cos^2(\frac{x}{2}) + \cos^2(\frac{x}{2}) = 2\cos^2\left(\frac{x}{2}\right) - 1.$$

Combining we have

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2} - 1 + 1} = \frac{\sin x}{\cos x + 1}.$$

Exercise 47: Let the function $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = \cos(z)\sin(z) - 1.$$

Find all $z \in \mathbb{C}$, for which $f(z) = 0$.

Solution:

$f(z) = 0 \Leftrightarrow \cos(z)\sin(z) = 1$. Thus we must find solutions to this equation with $z \in \mathbb{C}$:

$$1 = \sin z \cos z = \frac{1}{2i}(e^{iz} - e^{-iz})\frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{4i}(e^{2iz} - e^{-2iz})$$
$$\Rightarrow e^{2iz} - 4i - e^{-2iz} = 0.$$

Wir substituieren $e^{2iz} = u$:

$$u - 4i - u^{-1} = 0 \Rightarrow u^2 - 4iu - 1 = 0 \Rightarrow (u-2i)^2 + 3 = 0 \Rightarrow (u-2i)^2 = -3.$$

We substitute $w := u - 2i$ and find $Re(w) = 0$, $Im(w) = \pm\sqrt{3} \Rightarrow w = \pm\sqrt{3}i$. Thus, $u = (2 \pm \sqrt{3})i$. For $z = x + iy$ we have:

$$e^{2iz} = (2\pm\sqrt{3})i \Rightarrow 2i(x+iy) = \ln\left((2\pm\sqrt{3})i\right) \Rightarrow -2y + 2xi = \ln(2\pm\sqrt{3}) + \left(\frac{\pi}{2} + 2k\pi\right)i \quad (k \in \mathbb{Z})$$
$$x = \frac{\pi}{4} + k\pi \quad (k \in \mathbb{Z}), \quad y = -\frac{\ln(2\pm\sqrt{3})}{2}.$$

Thus $f(z) = 0$ for z of the form:

$$z = \frac{\pi}{4} + k\pi - \frac{\ln(2\pm\sqrt{3})}{2}i \quad (k \in \mathbb{Z}).$$

Exercise 48:

Find the real and the imaginary part of all complex numbers $z \in \mathbb{C}$, which satisfy the equation

$$\frac{e^{iz} - 1}{1 - 2e^{-iz}} = 1\,.$$

Solution:

Rearranging we have,

$$e^{iz} - 1 = -2\,e^{-iz} + 1$$

equivalently,

$$e^{iz} - 2 + 2\,e^{-iz} = 0$$

so

$$(e^{iz})^2 - 2\,e^{iz} + 2 = 0\,.$$

We substitute $u = e^{iz}$ to obtain the quadratic:

$$u^2 - 2u + 2 = 0\,.$$

Solving this quadratic $(u-1)^2 + 1 = 0$we obtain the solution

$$u_{1,2} = 1 \pm i\,.$$

So

$$e^{-y}(\cos x + i \sin x) = e^{iz} = 1 \pm i\,.$$

Write $z = x + iy$ where $x, y \in \mathbb{R}$. We have $e^{-y} = |1+i| = \sqrt{2}$, so $y = -\ln(\sqrt{2})$. It follows that one solution is $\cos x = \sin x = \frac{1}{\sqrt{2}}$ and so $x = \frac{\pi}{4} + 2\pi n$ where $n \in \mathbb{Z}$. The other solution is $\cos x = -\sin x = \frac{1}{\sqrt{2}}$ where $x = -\frac{\pi}{4} + 2\pi n$ and $n \in \mathbb{Z}$. Thus,

$$z = \pm\frac{\pi}{4} + 2\pi n - i \ln\sqrt{2}$$

where $n \in \mathbb{Z}$, is the set of all complex solutions.

Exercise 49:

(a) Find the real and imaginary parts of the numbers $i^i, \ln(i), \cos(i)$ and $e^{i\sqrt{2}}$.

(b) Show that the equation

$$\ln(u^v) = v\ln(u) \qquad \text{for } u, v \in \mathbb{C}$$

is not always satisfied.

Solution:

(a) Using the definition $a^z = e^{z\ln(a)}$ we deduce

$$i^i = e^{i\ln(i)} = e^{ii\frac{\pi}{2}} = e^{-\frac{\pi}{2}}.$$

Recall that the complex natural logarithm of $z = re^{i\varphi}$ is given by $\ln(z) = \ln(r) + i\varphi$. Writing $i = 1 \cdot e^{i\frac{\pi}{2}}$ we obtain $\ln(i) = \ln(1) + i\frac{\pi}{2} = i\frac{\pi}{2}$.
Using the formula $\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$ we have

$$\cos(i) = \frac{1}{2}(e^{-1} + e) \approx 1,54.$$

Since $e^{it} = \cos(t) + i\sin(t)$ we have

$$e^{\sqrt{2}} = \cos(\sqrt{2}) + i\sin(\sqrt{2}) \approx 0,16 + 0,99i.$$

(b) Using the identity $e^{2\pi i} = 1$ we find that

$$0 = \ln(1) = \ln(e^{2\pi i}) \text{ und } 2\pi i\ln(e) = 2\pi i.$$

Exercise 50: The power a^x is given by $a^x = e^{x\cdot\ln a}$ for $a, x \in \mathbb{R}$, $a > 0$. For some fixed a prove that

(a) $(a^x)^y = a^{xy}$ for all $x, y \in \mathbb{R}$,

(b) a^x is strictly monotonically increasing if $a > 1$, and strictly monotonically decreasing if $0 < a < 1$.

For $a = 10$ the inverse function $f^{-1}(y) = \log_{10}(y)$ of $f(x) = 10^x$ is the logarithm with respect to the basis 10.

(c) How can one use the logarithm $\ln(x)$ to compute the value $\log_{10}(x)$?

(d) Prove that
$$\begin{aligned} \log_{10}(xy) &= \log_{10}(x) + \log_{10}(y) \quad , \quad x, y > 0\,, \\ \log_{10}(x^y) &= y\log_{10}(x) \quad , \quad x > 0,\ y \in \mathbb{R}\,. \end{aligned}$$

Solution:

(a) $(a^x)^y = e^{y \ln a^x} = e^{y \ln e^{x \ln a}} = e^{yx \ln a} = a^{xy}$.

(b) Let $x > y$. For $a > 1$ we have $\ln a > 0$, so $x \ln a > y \ln a$. By monotonicity of the exponential we obtain $a^x = e^{x \ln a} > e^{y \ln a} = a^y$. For $0 < a < 1$ we have $\ln a < 0$, and so $x \ln a < y \ln a$.

(c) We have $x = 10^{\log_{10} x} = e^{\log_{10} x \ln 10}$. Thus by the change of base formula: $\ln x = \log_{10} x \ln 10$. We obtain the expression:

$$\log_{10} x = \frac{\ln x}{\ln 10}.$$

(d) For the first equation we use the formula from part (c) and the corresponding properties of the natural logarithm. For the second we set $z = \log_{10} x$, so $x = 10^z$. Then it follows that $x^y = 10^{yz}$, and we obtain

$$\log_{10} x^y = \log_{10} 10^{yz} = yz = y \log_{10} x.$$

Due date: Your written solutions are due at 14:00 on Tuesday, **15 January, 2019**.
Please submit them at the beginning of the problem session.

Website: For detailed information regarding this course visit the following web page:

`http://www.math.kit.edu/iag6/edu/am12018w/en`