Karlsruhe Institute of Technology
Institute for Algebra und Geometry
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Karlsruhe, December 20, 2019

Name: ....

Exercise Sheet No. 11
Advanced Mathematics I

Exercise 51:

(a) Find the derivatives of

(i) $f_1(x) = x^x, \quad x \in \mathbb{R}_{>0},$ (ii) $f_2(x) = (\sqrt{x}+1)\left(\frac{1}{\sqrt{x}} - 1\right), \quad x \in \mathbb{R}_{>0},$

(iii) $f_3(x) = \frac{\sin x}{\sin x + \cos x}, \quad x \in \left[0, \frac{\pi}{2}\right]$ (iv) $f_4(x) = e^{(\sin x)^2} + e^{\sin(x^2)} + \left(e^{\sin x}\right)^2, \quad x \in \mathbb{R}.$

(b) Show that the function defined by $f(x) = x^a \sin(\frac{1}{x})$ for $x > 0$ and $f(0) = 0$ where $x \in \mathbb{R}_{\geq 0}$ is continuously differentiable when $a = 3$ and not differentiable when $a = 1$.

Solution:

(a)

$$\text{(i) } f_1(x) = e^{x \ln x} \Rightarrow f_1'(x) = e^{x \ln x}(\ln x + 1) = x^x(\ln x + 1)$$

$$\text{(ii) } f_2'(x) = \left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)' = -\frac{1}{2x^{3/2}} - \frac{1}{2x^{1/2}} = -\frac{1+x}{2x^{3/2}}.$$

$$\text{(iii) } f_3'(x) = \frac{\cos x(\sin x + \cos x) - \sin x(\cos x - \sin x)}{(\sin x + \cos x)^2} = \frac{\cos^2 x + \sin^2 x}{(\sin x + \cos x)^2} = \frac{1}{(\sin x + \cos x)^2}.$$

$$\text{(iv) } f_4'(x) = e^{(\sin x)^2} 2 \sin x \cos x + e^{\sin x^2} \cos x^2 \, 2x + 2e^{2 \sin x} \cos x.$$

(b) We will use the definition of the derivative. For $a = 1$ we have

$$\lim_{x \to 0} \frac{x \sin\left(\frac{1}{x}\right) - 0}{x} = \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

and this limit does not exist. It follows that the function is not differentiable for $a = 1$.

For $a = 3$ we have

$$\lim_{x \to 0} \left| \frac{x^3 \sin\left(\frac{1}{x}\right) - 0}{x} \right| = \lim_{x \to 0} \left| x^2 \sin\left(\frac{1}{x}\right) \right| \leq \lim_{x \to 0} x^2 = 0$$

and so $f'(0) = 0$. For $x > 0$ we differentiate to obtain

$$|f'(x)| = \left| 3x^2 \sin\left(\frac{1}{x}\right) + x^3 \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) \right| = \left| 3x^2 \sin\left(\frac{1}{x}\right) - x \cos\left(\frac{1}{x}\right) \right| \leq |3x + x| = |4x| \to 0$$

as $x \to 0$. It follows that the function is continuously differentiable at 0 (and by the formula for $f'(x)$ it is easy to see it is continuously differentiated everywhere else) .

Exercise 52: Consider the function

$$f(x) = \begin{cases} e^x, & \text{if } x \leq 0, \\ \cos(x) + x, & \text{if } x > 0. \end{cases}$$

(a) Show that f is continuously differentiable at $x \neq 0$ arbitrarily many times.

(b) Show that f is (once) continuously differentiable at $x = 0$.

(c) Is f twice continuously differentiable at $x = 0$?

Solution:

(a) The functions e^x and $x + \cos x$ are continuously differentiable on $\mathbb{R}$ infinitely many times and the given piecewise defined function agrees with one of these functions on open intervals not containing 0.

(b) For $x \neq 0$

$$f'(x) = \begin{cases} e^x, & x < 0, \\ 1 - \sin x, & 0 < x. \end{cases}$$

and so

$$\lim_{x \to -0} f'(x) = 1 = \lim_{x \to +0} f'(x).$$

Thus f' has a limit of 1 as x tends to 0, which is equal to $f'(0)$. It follows that f is continuously differentiable at 0.

(c) For $x \neq 0$ we have

$$f''(x) = \begin{cases} e^x, & x < 0, \\ -\cos x, & 0 < x. \end{cases}$$

and it follows that

$$\lim_{x \to -0} f''(x) = 1 \neq -1 = \lim_{x \to +0} f''(x),$$

so f is not twice continuously differentiable at 0.

Exercise 53:

(a) Find all $x \in \mathbb{R}$, for which the function $f(x) = \sinh(x)$ is strongly monotone increasing (and thus has an inverse).

(b) Use the expression $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$, to find the inverse function $f^{-1}(x) = \text{Arcsinh}(x)$ and determine $(f^{-1})'(x)$.

Solution:

(a) We have $(\sinh x)' = \cosh(x) > 0$ for all $x \in \mathbb{R}$ and so f is strongly monotone and thus invertible for $\mathbb{R}$.

(b) Let $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$. We have:

$$2y = e^x - e^{-x} \iff e^{2x} - 2ye^x - 1 = 0 \iff$$
$$(e^x - y)^2 = 1 + y^2 \iff e^x = y \pm \sqrt{1 + y^2}$$

Since $e^x > 0$ and $y - \sqrt{1+y^2} < y - \sqrt{y^2} = 0$ it follows $e^x = y + \sqrt{1+y^2}$. Thus,

$$x = \ln(y + \sqrt{1+y^2}) \quad \text{hence.} \quad f^{-1}(x) = \ln(x + \sqrt{x^2+1}) = \text{Arsinh}(x).$$

Differentiating we obtain

$$\begin{aligned} \left(f^{-1}(y)\right)' &= \frac{1}{y + \sqrt{y^2+1}}(y + \sqrt{y^2+1})' = \frac{1 + \frac{1}{2}(y^2+1)^{-\frac{1}{2}}2y}{y + \sqrt{y^2+1}} \\ &= \frac{\sqrt{y^2+1} + y}{(y + \sqrt{y^2+1})\sqrt{y^2+1}} = \frac{1}{\sqrt{y^2+1}}. \end{aligned}$$

Exercise 54:

Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{x^2}{\sqrt{1+x^2}}.$$

Find the tangent line to f at the points $x_0 = 0, 1$ and $\sqrt{2}$. Show that the slope of the function $f(x)$ tends to 1 as $x \to \infty$.

Solution:

The tangent line of f at x_0 is given by $g(x) = f'(x_0)(x - x_0) + f(x_0)$. Calculating the derivative of f we obtain,

$$\begin{aligned} f'(x) &= \frac{2x\sqrt{1+x^2} - x^2\frac{1}{2}(1+x^2)^{-\frac{1}{2}}2x}{1+x^2} = \frac{2x\left((1+x^2)^{\frac{1}{2}} - \frac{x^2}{2}(1+x^2)^{-\frac{1}{2}}\right)}{1+x^2} \\ &= \frac{2x(1+x^2-\frac{x^2}{2})}{(1+x^2)(1+x^2)^{\frac{1}{2}}} = \frac{2x+x^3}{(1+x^2)^{\frac{3}{2}}}. \end{aligned}$$

Thus, the tangent line has the form

$$g(x) = \frac{2x_+x_0^3}{(1+x_0^2)^{\frac{3}{2}}}(x - x_0) + \frac{x_0^2}{\sqrt{1+x_0^2}}$$

and plugging in the given values of x yields,

$$g(x) = \begin{cases} 0, & x_0 = 0, \\ \frac{3}{\sqrt{8}}(x-1) + \frac{1}{\sqrt{2}}, & x_0 = 1 \\ \frac{4\sqrt{2}}{3\sqrt{3}}(x - \sqrt{2}) + \frac{2\sqrt{3}}{3}, & x_0 = \sqrt{2} \end{cases}$$

Taking the limit of f' as x tends to infinity we obtain 1 as required.

Exercise 55:

Prove that for $x \geq e^2$ that the inequality

$$\sqrt{x} > \ln(x)$$

holds.

Hint: Consider the function $f(x) = \sqrt{x} - \ln x$.

Solution:

First consider $x = e^2$. We have $\sqrt{x} = \sqrt{e^2} = e > \ln e^2 = 2 \ln e = 2$.

Define the function $f(x) = \sqrt{x} - \ln(x)$ for $x \geq e^2$.. Differentiating we obtain

$$f'(x) = \frac{1/2}{\sqrt{x}} - \frac{1}{x} > \frac{1/2}{\sqrt{e^2}} - 1/x > 1 - \frac{1}{x} > 0$$

. Since $f'(x)$ is positive on the domain of f, it follows that f is monotone increasing. Since $f(e^2) > 0$ it follows that $\sqrt{x} - \ln(x) > 0$ for $x \geq e^2$ and so $\sqrt{x} > \ln(x)$, as required.

Due date: Your written solutions are due at 14:00 on Tuesday, **22 January, 2018**. Please submit them at the beginning of the problem session.

Website: For detailed information regarding this course visit the following web page:

`http://www.math.kit.edu/iag6/edu/am12018w/en`