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Karlsruhe, Jan 17, 2019

Name: ....

Exercise Sheet No. 12
Advanced Mathematics I

Exercise 56: Compute the following limits:

$$\text{(a)}\quad \lim_{x\to 0}\left(\frac{1}{x^3+ax^2+x}-\frac{1}{\sin x}\right),\qquad \text{(b)}\quad \lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x},\qquad \text{(c)}\quad \lim_{x\to 0}(\cos x)^{1/x^2}.$$

The coefficient $a\in\mathbb{R}$ in part (a) is a constant.

Solution:

We will apply l'Hospital's rule to evaluate these limits.

(a) We have

$$\frac{1}{x^3+ax^2+x}-\frac{1}{\sin x}=\frac{\sin x-(x^3+ax^2+x)}{(x^3+ax^2+x)\sin x}.$$

By l'Hospital's rule we have

$$\lim_{x\to 0}\left(\frac{1}{x^3+ax^2+x}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\frac{\cos x-3x^2-2ax-1}{(x^3+ax^2+x)\cos x+(3x^2+2ax+1)\sin x}.$$

Applying l'Hospital a second time we get

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x^3+ax^2+x}-\frac{1}{\sin x}\right)&=\lim_{x\to 0}\frac{-\sin x-6x-2a}{(6x^2+4ax+2)\cos x-(x^3+ax^2-5x-2a)\sin x}\\&=\frac{-2a}{2}=-a.\end{aligned}$$

(b) Rewriting yields

$$\left(\frac{1}{x}\right)^{\tan x}=e^{\tan x\,\ln\frac{1}{x}}=e^{-\tan x\,\ln x}=\exp\left(-\frac{\tan x}{\frac{1}{\ln x}}\right).$$

We may bring the limit inside the exponential function by continuity, and the function inside has the $0/0$ form required to apply l'Hospital's rule.

$$\lim_{x\to 0}\frac{\tan x}{\frac{1}{\ln x}}=\lim_{x\to 0}\frac{\frac{1}{\cos^2 x}}{-\frac{1}{x(\ln x)^2}}=\lim_{x\to 0}-\left(\frac{\sqrt{x}\ln x}{\cos x}\right)^2$$

Evaluating the resulting limit inside the exponential we obtain

$$\lim_{x\to 0}\left(\sqrt{x}\ln x\right)=\lim_{x\to 0}\frac{\ln x}{\frac{1}{\sqrt{x}}}\overset{\frac{-\infty}{\infty}}{=}\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{2x^{3/2}}}=\lim_{x\to 0}-2\sqrt{x}=0.$$

Thus, the original limit becomes

$$\lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}=\exp\left(\lim_{x\to 0}\left(\frac{\sqrt{x}\ln x}{\cos x}\right)^2\right)=\exp\left(\left(\frac{0}{1}\right)^2\right)=1.$$

(c) We will again rewrite the expression using the exponential function and apply l'Hospital's rule:

$$\begin{aligned}\lim_{x\to 0}(\cos x)^{1/x^2}&=\lim_{x\to 0}\exp\left(\frac{\ln\cos x}{x^2}\right)=\exp\left(\lim_{x\to 0}\frac{\ln\cos x}{x^2}\right)\\&\overset{\frac{0}{0}}{=}\exp\left(\lim_{x\to 0}\frac{\frac{1}{\cos x}(-\sin x)}{2x}\right)=\exp\left(-\lim_{x\to 0}\frac{\sin x}{2x\cos x}\right)\\&\overset{\frac{0}{0}}{=}\exp\left(-\lim_{x\to 0}\frac{\cos x}{2\cos x-2x\sin x}\right)=e^{-1/2}.\end{aligned}$$

Exercise 57:

Determine the constant $c\in\mathbb{R}$ such that the function $f(x)=\begin{cases}c, & x=1,\\ \frac{2^{\ln x}-x}{\ln x}, & x\neq 1,\end{cases}$ on $\mathbb{R}_{>0}$ is continuous.

Solution:

For continuity to hold we must have,

$$c = f(1) \stackrel{!}{=} \lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{2^{\ln x} - x}{\ln x}.$$

Rewriting this expression with the exponential function we obtain:

$$\frac{2^{\ln x} - x}{\ln x} = \frac{e^{\ln 2\, \ln x} - x}{\ln x} = \frac{x^{\ln 2} - x}{\ln x}.$$

We have an expression with the 0/0 form and apply l'Hospital's rule:

$$\lim_{x\to 1} \frac{x^{\ln 2} - x}{\ln x} = \lim_{x\to 1} \frac{\ln 2\, x^{\ln 2 - 1} - 1}{\frac{1}{x}} = \ln 2 - 1.$$

Thus we need $c = \ln 2 - 1$ for f defined on $\mathbb{R}_{>0}$ to be continuous.

Exercise 58: Calculate all the derivatives $f^{(n)}$, $n = 0, 1, 2, \ldots$ of the function f and give the Taylor series for f with center of expansion $x_0 = 0$. Where does the series in part (a) converge?

$$\text{(a)} \quad f(x) = \cosh\frac{x}{2}, \quad x \in \mathbb{R}, \qquad\qquad \text{(b)} \quad f(x) = \sqrt{1+x}, \quad |x| \le 1.$$

Hint for (b): The derivatives of f have the following form: $f^{(k)}(x) = -(-1)^k \frac{(2k-2)!}{2^{2k-1}\,(k-1)!}(1+x)^{-\frac{2k-1}{2}}$.

Solution:

(a) By considering the values $k = 1, 2, 3$ we guess the following form for $f^{(k)}$:

$$f^{(k)}(x) = \begin{cases} \frac{1}{2^k}\sinh\left(\frac{x}{2}\right), & k \text{ ungerade} \\ \frac{1}{2^k}\cosh\left(\frac{x}{2}\right), & k \text{ gerade} \end{cases}$$

We prove the above form is correct by induction.

- $k = 1$: $f'(x) = \frac{1}{2}\sinh(\frac{x}{2})$ ✓
- $k = 2$: $f''(x) = \frac{1}{4}\cosh(\frac{x}{2})$ ✓
- $k \to k+1$: 1. For k odd: $f^{(k+1)}(x) = \left(\frac{1}{2^k}\sinh(\frac{x}{2})\right)' = \frac{1}{2^{k+1}}\cosh(\frac{x}{2})$
  2.For k even $f^{(k+1)}(x) = \left(\frac{1}{2^k}\cosh(\frac{x}{2})\right)' = \frac{1}{2^{k+1}}\sinh(\frac{x}{2})$ □

We now find the Taylor expansion. Observe $\sinh(0) = 0$ and $\cosh(0) = 1$, so we have $f^{(2k+1)}(x) = 0$ and $f^{(2k)}(x) = \frac{1}{2^{2k}}$. We obtain the series:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!}(x-0)^{2k} = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}(2k)!}.$$

We analyze convergence with the ratio test. Set $a_k = \frac{x^{2k}}{2^{2k}(2k)!}$, then

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{2^{2k}(2k)!|x|^{2k+2}}{2^{2k+2}(2k+2)!|x|^{2k}} = \frac{|x|^2}{4(2k+2)(2k+1)} \to 0, \quad (k\to\infty)$$

Thus, the series converges on $\mathbb{R}$.

(b)

(i) We will prove the formula for the kth derivative by induction:

- $k = 1$: We have $f(x) = (x+1)^{1/2}$ so $f'(x) = \frac{1}{2}(x+1)^{-1/2}$. It follows that

$$f^{(1)}(x) = -(-1)^1 \frac{(2-2)!}{2^{2-1}\,0!}(1+x)^{-\frac{2-1}{2}} = \frac{1}{2}(1+x)^{-1/2} = f'(x)$$

- $k \to k+1$: We assume the result holds for k and prove it for $k+1$. We have

Induction Hypothesis $\quad f^{(k)}(x) = -(-1)^k \dfrac{(2k-2)!}{2^{2k-1}\,(k-1)!}(1+x)^{-\frac{2k-1}{2}}$

Induction Step $\quad \left(f^{(k)}\right)'(x) = f^{(k+1)}(x) = (-1)^k \dfrac{(2k)!}{2^{2k+1}\,k!}(1+x)^{-\frac{2k+1}{2}}$

Thus we obtain for $f^{(k)}$:

$$\begin{aligned}\left(f^{(k)}\right)'(x) &= -(-1)^k \frac{(2k-2)!}{2^{2k-1}\,(k-1)!}\left(-\frac{2k-1}{2}\right)(1+x)^{-\frac{2k+1}{2}} \\ &= (-1)^k \frac{(2k-2)!(2k-1)}{2^{2k-1}(k-1)!\cdot 2}(1+x)^{-\frac{2k+1}{2}} \\ &= (-1)^k \frac{(2k-1)!}{2^{2k}(k-1)!}(1+x)^{-\frac{2k+1}{2}} \\ &= (-1)^k \frac{(2k)!\frac{1}{2k}}{\frac{1}{2}2^{2k+1}\frac{1}{k}k!}(1+x)^{-\frac{2k+1}{2}} \\ &= (-1)^k \frac{(2k)!}{2^{2k+1}k!}(1+x)^{-\frac{2k+1}{2}} = f^{(k+1)}(x)\end{aligned}$$

as we were required to prove.

(ii) The Taylor Series at x_0 has the form

$$\left(\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n\right)$$

and for $x_0 = 0$ we obtain:

$$f^{(n)}(0) = -(-1)^n \frac{(2n-2)!}{2^{2n-1}\,(n-1)!}$$

Since $f(0) = \sqrt{0+1} = 1$, we have:

$$\left(1 - \sum_{n=1}^{\infty} \frac{(-1)^n(2n-2)!}{2^{2n-1}\,(n-1)!\,n!}x^n\right)$$

Exercise 59:

(a) Determine the Taylor formula for $m = 2$ about the point $x_0 = 8$ for the function $f(x) = x^{2/3}$, $x \geq 1$, find an expression for the Lagrange form of the remainder.

(b) For natural numbers n and real x, $1 + x > 0$, show using the Taylor formula that

$$(1+x)^n \geq 1 + nx.$$

Solution:

(a) $f(x) = x^{\frac{2}{3}}, f(x_0) = 8^{\frac{2}{3}} = 4; f'(x) = \frac{2}{3}x^{\frac{-1}{3}}, f'(x_0) = \frac{1}{3}; f''(x) = \frac{-2}{9}x^{\frac{-4}{3}}, f''(x_0) = -\frac{1}{72}; f'''(x) = \frac{8}{27}x^{\frac{-7}{3}}$.
We obtain $f(x) = 4 + \frac{1}{3}(x-8) - \frac{1}{2!}\frac{1}{72}(x-8)^2 + \frac{1}{3!}\frac{8}{27}\xi^{\frac{-7}{3}}(x-8)^3$ for some ξ between x and $x_0 = 8$.
We now find bounds on $|f'''(\xi)|$, where ξ is between x and $x_0 = 8$.
For $\xi > \min(x,\ 8) \geq 1$ we have $\frac{8}{27}\xi^{\frac{-7}{3}} \leq \frac{8}{27}$. Thus, $|R_2(x,\ \xi)| \leq \frac{1}{3!}\frac{8}{27}|x-8|^3 = \frac{4}{81}|x-8|^3$.

(b) For $n = 1$ this is obvious and for $n \geq 2$ we defined $f(x)$ to be the right-hand-side of the given expression. It follows that

$$f'(x) = n\,(1+x)^{n-1}, \quad f''(x) = n\,(n-1)\,(1+x)^{n-2}.$$

The Taylor expansion of f about 0 is given by

$$f(x) = f(0) + f'(0)\,x + f''(\xi)\,\frac{x^2}{2} = 1 + nx + n\,(n-1)\,(1+\xi)^{n-2}\frac{x^2}{2}$$

where ξ is between 0 and x. Since $\xi > -1$, the term $1+\xi$ is nonnegative and so $n(n-1)(1+\xi)^{n-2} \geq 0$. It follows that

$$(1+x)^n = 1 + nx + n\,(n-1)\,(1+\xi)^{n-2}\,\frac{x^2}{2} \geq 1 + nx,$$

as required.

Exercise 60:

(a) Define the function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 1$ for $x \geq 0$ and $f(x) = -1$ for $x < 0$. Show that the function $F(x) = \int_0^x f(t)dt$ is not a primitive (antiderivative) of f.

(b) Given $g\colon \mathbb{R}_{>0} \to \mathbb{R}$ defined by $g(x) = \int_{2\pi}^{x} \frac{\sin(t)}{t}dt$. Find the Taylor polynomial $p_2(x)$ of degree 2 about $x_0 = 2\pi$.

Solution:

(a) For $x \geq 0$ we have $F(x) = \int_0^x 1dx = x$ and for $x < 0$ we have $F(x) = \int_0^x (-1)dx = -x$, so $F(x) = |x|$. This function is not differentiable at $x = 0$ and so cannot be a primitive.

(b) Using the fundamental theorem of Calculus we obtain $g'(x) = \frac{\sin(x)}{x}$. It follows that

$$g''(x) = \left(\frac{\sin(x)}{x}\right)' = \frac{\cos(x)x - \sin(x)}{x^2}.$$

Plugging in x_0 yeilds, $g(2\pi) = 0$, $g'(2\pi) = 0$ and $g''(2\pi) = \frac{1}{2\pi}$. The Taylor polynomial is hence

$$p_2(x) = \frac{f(2\pi)}{0!} + \frac{f'(2\pi)}{1!}(x - 2\pi) + \frac{f''(2\pi)}{2!}(x - 2\pi)^2 = \frac{1}{4\pi}(x - 2\pi)^2.$$

Due date: Your written solutions are due at 14:00 on Tuesday, **29 January, 2019**.
Please submit them at the beginning of the problem session.

Website: For detailed information regarding this course visit the following web page:

`http://www.math.kit.edu/iag6/edu/am12018w/en`