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Karlsruhe, November 16, 2018

Name: ....

Exercise Sheet No. 3
Advanced Mathematics I

Exercise 11: Consider the complex numbers $z_1 = 1 + i, z_2 = 2 - 3i, z_3 = \sqrt{3} + i$.

(a) Find the real and imaginary part of the numbers $\bar{z_j} - z_j, z_j\bar{z_j}, \frac{1}{z_j}, z_j - \bar{z_j}$ and $|z_j|$ for $j = 1, 2$.

(b) Find a representation of z_3 in polar coordinates (r, ϕ).

Solution: (a) First observe $\bar{z_1} = 1 - i$, $\bar{z_2} = 2 + 3i$. Then we have $\bar{z_1} - z_1 = -2i$, $\bar{z_2} - z_2 = 6i$, $z_1\bar{z_1} = 2$, $z_2\bar{z_2} = 13$
$\frac{1}{z_1} = \frac{\bar{z_1}}{z_1\bar{z_1}} = \frac{1}{2} - \frac{1}{2}i$, $\frac{1}{z_2} = \frac{\bar{z_2}}{z_2\bar{z_2}} = \frac{2}{13} + \frac{3}{13}i$, $z_1 - \bar{z_1} = 2i$,$z_1 - \bar{z_1} = -6i$, $|z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$, $|z_2| = \sqrt{2^2 + 3^2} = \sqrt{13}$.

(b) We have $r = |z_3| = \sqrt{3 + 1} = 2$ and $\phi = \arctan(\frac{1}{\sqrt{3}}) = \pi/6$ Thus $z_3 = re^{i\phi} = 2e^{i\pi/6}$.

Exercise 12: Let $z \in \mathcal{C} \setminus \{-i\}$ and define w as follows:

$$w = \frac{(\overline{2 - i})z - 1 + 2i}{z + i}.$$

Compute the the real and imaginary part of w as a function of z, and also express w in polar form.

Solution: Let $z = x + iy$ with $x, y \in \mathbb{R}$ and $(x, y) \neq (0, -1)$. We will multiply and divide the given formula by the complex conjugate of the denominator:

$$\begin{aligned} w &= \frac{\left[(\overline{2 - i})z - 1 + 2i\right][\bar{z} - i]}{[z + i][\bar{z} - i]} = \frac{[(2 + i)(x + iy) - 1 + 2i][x - iy - i]}{[x + iy + i][x - iy - i]} \\ &= \frac{[(2x - y - 1) + i(x + 2y + 2)][x - i(y + 1)]}{x^2 + (y + 1)^2} \\ &= \frac{\left(2x^2 - xy - x + [(x + 2y + 2)(y + 1)]\right)}{x^2 + (y + 1)^2} + i\frac{x^2 + 2xy + 2x - (2x - y - 1)(y + 1)}{x^2 + (y + 1)^2} \\ &= \frac{2x^2 + 2y^2 + 4y + 2}{x^2 + (y + 1)^2} + i\frac{x^2 + y^2 + 2y + 1}{x^2 + (y + 1)^2} = 2 + i. \end{aligned}$$

It follows that

$$\operatorname{Re} w = 2, \ \operatorname{Im} w = 1, \ |w| = \sqrt{5}, \ \operatorname{Arg} w = \arctan\frac{1}{2}.$$

Remark Alternatively, we could do the following which is a bit faster:

$$\begin{aligned} w &= \frac{(2 + i)z - 1 + 2i}{z + i} = \frac{(2 + i)z + i(2 + i) - i(2 + i) - 1 + 2i}{z + i} = \frac{(2 + i)(z + i)}{z + i} + \frac{-i(2 + i) - 1 + 2i}{z + i} \\ &= 2 + i + \frac{-2i + 1 - 1 + 2i}{z + i} = 2 + i + \frac{0}{z + i} = 2 + i. \end{aligned}$$

Exercise 13: Sketch the following regions of $\mathbb{C}$:

$$\begin{aligned} K &= \{z \in \mathbb{C} : |2z - 4i|^2 = 16\}, \\ R &= \{z \in \mathbb{C} : 1 \leq |z - 3 + 4i| \leq 3\}, \\ G &= \{z \in \mathbb{C} : |z - 1 + i| = |z - 2 - i|\}, \\ H &= \{z \in \mathbb{C} : Re(z(1 - i)) \geq 0\}. \end{aligned}$$

Solution: K: Since the complex absolute value always yields a positive number, we may take a square root of both sides and divide by 2 to obtain $|z - 2i| = 2$. Thus, the region is a circle of radius 2 centered at $2i$.

R: The region is a closed annulus centered at $3 - 4i$ with inner radius 1 and outer radius 3.

G: The equation implies that the distance of z from $1 - i$ is equal to the distance from $2 + i$. Thus if we take the line segment from $1 - i$ to $2 + i$, then the region is the perpendicular line through the midpoint $(1/2)(1 - i) + (1/2)(2 + i) = 3/2$.

H: Let $z = a + bi$. Then $Re((a + bi)(1 - i)) = a + b$. We require $a + b \geq 0$ and so $b \geq -a$. Thus, the region is those points in the complex plane above and to the right of a line with slope -1 though the origin, including the line itself.

Exercise 14: Solve the following quadratic equations over the complex numbers.

(a)

$$z^2 + (-10 + 4i)z + 70 - 20i = 0,$$

(b)

$$z^2 + 6z - 3 + i(4z + 6) = 0.$$

Solution: (a) We ill write the solution in the form $z^2 + 2uz + v = 0$. We have $v = 70 - i20$ and $2u = -10 + i4$. It follows that $u = -5 + i2$. Equations in the form above can be rewritten as follows:

$$0 = z^2 + 2uz + v = z^2 + 2uz + u^2 - u^2 + v = (z + u)^2 - (u^2 - v)$$

Now we calculate $u^2 - v$:

$$u^2 = (-5)^2 - 2^2 + i2(-5)2 = 21 - i20$$
$$u^2 - v = 21 - i20 - 70 + i20 = -49$$

Rearranging yields,

$$(z + u)^2 = -49.$$

Solving this simple quadratic we obtain $z + u = i7$ or $z + u = -i7$, from which solving for z yields the solutions,

$$z = -u + i7 = 5 + i5, \qquad z = -u - i7 = 5 - i9\,.$$

(b) Again, we will put the equation in the form $z^2 + 2uz + v = 0$. We find $v = -3 + i6$ and $u = 3 + i2$, and thus $u^2 = 5 + 12i$ and $u^2 - v = 8 + i6$. The equation $z^2 - 6z + 3 + i(2 - 4z) = 0$ is equivalent to

$$(z + u)^2 = 8 + i6$$

Let $a = \mathrm{Re}(z + u)$ and $b = \mathrm{Im}(z + u)$, so $z + u = a + ib$. Then, $(z + u)^2 = a^2 - b^2 + i2ab$, from which we obtain,

$$a^2 - b^2 = 8 = \mathrm{Re}(8 + i6),$$
$$ab = 3 = \tfrac{1}{2}\,\mathrm{Im}(8 + i6) \qquad \Longleftrightarrow \qquad b = \tfrac{3}{a}.$$

From here we can solve for a,

$$a^2 - \frac{9}{a^2} - 8 = 0 \qquad \Longleftrightarrow \qquad 0 = (a^2)^2 - 8a^2 - 9 = (a^2)^2 - 2 \cdot 4a^2 + 16 - 16 - 9 = (a^2 - 4)^2 - 25\,.$$

We obtain $(a^2 - 4)^2 = 25$ from which we have $a^2 = 4 + 5$ or $a^2 = 4 - 5$. Since a was chosen to be real it follows that $a^2 = 9$.
Fort $a = \sqrt{9} = 3$ we have $b = \frac{3}{3} = 1$ and for $a = -3$ we have $b = -1$. Thus, we obtain the solutions,

$$z = -3 - i2 + 3 + i = -i, \qquad z = -6 - i3\,.$$

Exercise 15: Show the following by induction (Note: $i = \sqrt{-1}$):

$$\sum_{k=0}^{2n} i^k k = \begin{cases} n(1 - i) \text{ when } n \text{ is even} \\ -(n + 1) + ni \text{ when } n \text{ is odd.} \end{cases}$$

Solution: Remark: There are two natural ways to proceed with a proof like this where the even and odd case of a formula are different. We can either proceed by induction on n and consider 2 cases at each step, or we can break the problem into two separate induction proofs, one for the odd case and one for the even.

We will do the first type of proof.

Base Case: For the base case observe

$$n = 0 \quad : \quad i^0 \cdot 0 = 0 \cdot (1 - i) \text{ true.}$$
$$n = 1 \quad : \quad \sum_{k=0}^{2} i^k k = 0 + i - 2 = -2 + i \text{ true.}$$

Induction Step: Suppose the result holds for smaller $n \in \mathbb{N}$. Consider

$$\sum_{k=0}^{2n+2} i^k k = \begin{cases} (n + 1)(1 - i), & \text{when } n + 1 \text{ is even, so } n \text{ is odd,} \\ -(n + 2) + (n + 1)i, & \text{when } n + 1 \text{ is odd, so } n \text{ is even.} \end{cases}$$

We separate out the last two summands in the sum,

$$\sum_{k=0}^{2n+2} i^k k = \left(\sum_{k=0}^{2n} i^k k\right) + i^{2n+1}(2n + 1) + i^{2n+2}(2n + 2)$$

Now we have to distinguish two cases:

(i) For n even, we use the first part of the induction hypothesis to obtain

$$\sum_{k=0}^{2n+2} i^k k \overset{\text{IH}}{=} n(1-i) + (2n+1)i - (2n+2) = (n+1)i - (n+2)$$

as required.

(ii) For n odd, we hae

$$\sum_{k=0}^{2n+2} i^k k \overset{\text{IH}}{=} -(n+1) + ni - (2n+1)i + (2n+2) = (n+1) - (n+1)i,$$

and the proof is complete.

Due date: Your written solutions are due at 14:00 on Tuesday, **13 November, 2018**.
Please submit them at the beginning of the problem session
or in the box in J101 (note the box will be emptied before the problem session).

Website: For detailed information regarding this course visit the following web page:

`http://www.math.kit.edu/iag6/edu/am12018w/en`