Karlsruhe Institute of Technology
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Karlsruhe, November 30, 2018

Name: ....

Exercise Sheet No. 7
Advanced Mathematics I

Exercise 31:

(a) Compute the following limits:

$$\text{(i)} \quad \lim_{x\to\infty} \frac{\sqrt{x^2+4}}{\sqrt{x^2+9}}, \qquad \text{(ii)} \quad \lim_{x\to\infty} x^\alpha(\sqrt{x+1}-\sqrt{x}), \quad \alpha \in \mathbb{R}.$$

(b) Find $\lim_{x\to x_0} f(x)$ for the following functions f at x_0:

$$\text{(i)} \quad f(x) = \frac{x-2}{x^2-4} \text{ for } x>2, \quad x_0 = 2, \qquad \text{(ii)} \quad f(x) = \frac{\sqrt[4]{x}-1}{\sqrt[3]{x}-1} \text{ for } x>1, \quad x_0 = 1.$$

Solution:

(a) (i) $$\lim_{x\to\infty} \frac{\sqrt{x^2+4}}{\sqrt{x^2+9}} = \lim_{x\to\infty} \frac{x\sqrt{1+\frac{4}{x^2}}}{x\sqrt{1+\frac{9}{x^2}}} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{4}{x^2}}}{\sqrt{1+\frac{9}{x^2}}} = 1.$$

(ii) $$x^\alpha(\sqrt{x+1}-\sqrt{x}) = \frac{x^\alpha}{\sqrt{x+1}+\sqrt{x}}(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x}) = \frac{x^\alpha}{\sqrt{x+1}+\sqrt{x}} = \frac{x^\alpha}{\sqrt{x}(\sqrt{1+\frac{1}{x}}+1)} = \frac{x^{\alpha-\frac{1}{2}}}{\sqrt{1+\frac{1}{x}}+1}.$$

Thus, $$\lim_{x\to\infty} x^\alpha(\sqrt{x+1}-\sqrt{x}) = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} \cdot \lim_{x\to\infty} x^{\alpha-\frac{1}{2}} = \begin{cases} 0 & : \alpha - 1/2 < 0 \\ 1/2 & : \alpha - 1/2 = 0 \\ +\infty & : \alpha - 1/2 > 0 \end{cases}.$$

(b) (i) For $x > 2$ we can write f as

$$f(x) = \frac{1}{x+2}. \qquad \text{and so } \lim_{x\to 2} f(x) = \frac{1}{4}.$$

(ii) We make use of the following identities which we obtain by simple factoring:

$$a^4 - b^4 = (a-b)\,(a^3 + a^2b + ab^2 + b^3)$$
$$c^3 - d^3 = (c-d)\,(c^2 + cd + d^2)$$

Applying these identities with the choices $a = \sqrt[4]{x} = x^{1/4}$, $b = 1$, $c = \sqrt[3]{x} = x^{1/3}$ and $d = 1$ yields,

$$\begin{aligned} f(x) &= \frac{(\sqrt[4]{x}-1)\,(x^{3/4}+x^{2/4}+x^{1/4}+1)\,(x^{2/3}+x^{1/3}+1)}{(\sqrt[3]{x}-1)\,(x^{3/4}+x^{2/4}+x^{1/4}+1)\,(x^{2/3}+x^{1/3}+1)} \\ &= \frac{(x-1)\,(x^{2/3}+x^{1/3}+1)}{(x-1)\,(x^{3/4}+x^{2/4}+x^{1/4}+1)} \\ &= \frac{x^{2/3}+x^{1/3}+1}{x^{3/4}+x^{2/4}+x^{1/4}+1}. \end{aligned}$$

Thus, we have

$$\lim_{x\to 1} f(x) = \frac{1+1+1}{1+1+1+1} = \frac{3}{4}.$$

Exercise 32:

(a) Determine whether the following sets are open, closed or compact. Justify your answers.

$$M_1 = [-1, 42] \subset \mathbb{R}, \quad M_2 = (-1, 42] \subset \mathbb{R}, \quad M_3 = \{z \in \mathbb{C} : -1 \le \operatorname{Im} z \le 1\} \subset \mathbb{C},$$
$$M_4 = \left\{z \in \mathbb{C} : \operatorname{Im} z \ge 0, |z - \frac{1}{2}| \le \frac{1}{2}\right\} \setminus \{0\} \subset \mathbb{C}.$$

(b) Show that $z = i$ is in the boundary of M_3 and $z = \frac{1}{4}$ is an interior point of M_4.

Solution:

(a) $\underline{M_1}$: Closed and bounded and thus compact. Not open since if we take an any open interval containing 42, say (a, b), we will always find an element outside of M_1 (for example, $\frac{b-42}{2}$).

$\underline{M_2}$: Bounded (everything in M_2 has norm at most 42). M_2 is not closed because the sequence $-1+1/n$ converges to $-1 \notin M_2$. M_2 is not open by the same reasoning as for M_1. Not compact since not both closed and bounded.

$\underline{M_3}$: Closed. Not bounded since M_3 contains all reals and reals can have arbitrarily large complex norm. Not compact since not bounded. Not open because an open ball about i always contains elements outside of M_3 (for example, if the ball has radius r, then $\frac{i+ri}{2}$ is not in M_3 but is in the ball.

$\underline{M_4}$: Bounded. Not closed, take any sequence converging to 0 in M_4. Not compact since not closed. Not open, consider an open ball at 1.

(b) To show that i is on the boundary of M_3, notice that $i - \frac{i}{n}$, gives a sequence of points in M_3 converging to i, and $i + \frac{i}{n}$ gives a sequence of points not in M_3, converging to i. In fact 1/4 is not an interior point! Indeed, take any open ball about 1/4, it will contain a number with negative imaginary part, and thus ouside of M_4.

Exercise 33:

Consider the following polynomial p and function f:

$$p(x) \;=\; x^5 - 8x^2 + 4\,, \quad f(x) = |p(x)|\;, \quad x \in \mathbb{R}.$$

(a) Justify why the function must have a minimum and maximum value on $[-2, 2]$. Let the minimum be attained at x_-, and show $f(x_-) \leq 4$.

(b) Show that $f(x) \geq 4$ for all $|x| \geq 2$.

(c) Does f defined on $\mathbb{R}$ have a minimum?

Solution:

(a) The interval $[-2, 2]$ is compact. The function f is continuous as the composition of continuous functions. Thus f must have a maximum and minimum on $[-2, 2]$

For $x = 2$ we have $f(x) = 4$. Thus the minimum can be at most 4.

(b) The polynomial p can be rewritten $p(x) = x^5 - 8x^2 + 4 = x^2(x^3 - 8) + 4$, so for $x \geq 2$

$$|f(x)| = |x^2(x^3 - 8) + 4| \geq 4,$$

since x^2 and $x^3 - 8$are positive. For $x \leq 2$ we have

$$|f(x)| = |x^2(x^3 - 8) + 4| = |x^2(x^3 - 8) - (-4)| \geq |x^2(x^3 - 8)| - |-4| = |x^2(x^3 - 8)| - 4.$$

x^2 is at least 4 and $(x^3 - 8) \leq -8 - 8 = -16 \Rightarrow -(x^3 - 8) \geq 16$. So we have $|x^2(x^3 - 8)| \geq 4 \cdot 16$. Thus,

$$|f(x)| \geq 4 \cdot 16 - 4 = 4 \cdot 15 = 60 \geq 4.$$

(c) Yes the function has a minimum. Outside of $[-2, 2]$ the function has value at least 4. Inside $[-2, 2]$ we know f attains a minimum and that minimum is at most 4. Thus f must attain a minimum on $\mathbb{R}$.

Exercise 34: How many solutions does the following equation have in the interval $I = [-4, 2]$?

$$x^5 + 8x^4 + 11x^3 = 25x^2 + 26x + 5$$

Justify your answer.
Hint: Formulate the equation in the form $f(x) = 0$ and evaluate f at the points $x = -4, -2, -\frac{1}{2}, 1$ and 2.

Solution:

Consider the polynomial

$$f(x) = x^5 + 8x^4 + 11x^3 - 25x^2 - 26x - 5.$$

We have the values

$$\begin{aligned} f(-4) &= 19 \\ f(-2) &= -45 \\ f(-\tfrac{1}{2}) &= \tfrac{27}{32} \end{aligned} \qquad \begin{aligned} f(1) &= -36 \\ f(2) &= 91. \end{aligned}$$

Thus by the intermediate value theorem, there must be zeros on the intervals $(-4, -2)$, $(-2, -\frac{1}{2})$, $(-\frac{1}{2}, 1)$ and $(1, 2)$. Since for $x \to -\infty$ we have that f tends to $-\infty$, there must be another zero smaller than -4. Thus there are at least 5 zeros. By the fundamental theorem of algebra there are at most 5 zeros, since our polynomial has degree 5. Every zero of our polynomial is a solution of the original equation, so we have 5 solutions.

Exercise 35: Show that for any positive constants a, b, c the equation

$$\frac{(a+b)x+a-b}{x^2-1}+\frac{c}{x-2}=1$$

has solutions in the intervals $[-1,1]$ and $[1,2]$, respectively.

Solution:

We have

$$\frac{(a+b)x+a-b}{x^2-1}+\frac{c}{x-2}=\frac{a}{x-1}+\frac{b}{x+1}+\frac{c}{x-2}.$$

So we have the limit

$$\lim_{x\searrow -1}\left(\frac{(a+b)x+a-b}{x^2-1}+\frac{c}{x-2}\right)=+\infty$$

so there is x_1 near -1 for which the expression is positive. Since

$$\lim_{x\nearrow 1}\left(\frac{(a+b)x+a-b}{x^2-1}+\frac{c}{x-2}\right)=-\infty.$$

there is x_2 near 1 for which the expression is negative. Thus, by the intermediate value theorem, there must be a solution in the interval $(-1,1)$. Similar reasoning applies for the interval $(1,2)$.

Due date: Your written solutions are due at 14:00 on Tuesday, **11 December, 2018**.
Please submit them at the beginning of the problem session.

Website: For detailed information regarding this course visit the following web page:

`http://www.math.kit.edu/iag6/edu/am12018w/en`