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Name: ....

Exercise Sheet No. 9
Advanced Mathematics I

Exercise 41: For each of the following power series find all $z \in \mathbb{C}$ where the series converges

$$\text{(a)} \quad \left(\sum_{n=0}^{\infty} \frac{z^n}{2(n!)}\right), \quad \text{(b)} \quad \left(\sum_{n=1}^{\infty} \frac{(z-1)^{2n}}{\left(1+\frac{1}{n}\right)^n}\right), \quad \text{(c)} \quad \left(\sum_{n=1}^{\infty} n^n(z+2)^n\right).$$

Solution: We denote the radius of convergence by r.
(a) Applying the quotient test yields:

$$\left|\frac{2(n!)}{2(n+1)!} \ \frac{z^{n+1}}{z^n}\right| = \frac{|z|}{n+1} \overset{n\to\infty}{\longrightarrow} 0 < 1 \text{ for all } z \quad \Rightarrow r = \infty$$

Thus, the series converge for all z.
(b) Applying the root test we obtain:

$$\sqrt[n]{\frac{|z-1|^{2n}}{(1+\frac{1}{n})^n}} = \frac{|z-1|^2}{1+\frac{1}{n}} \overset{n\to\infty}{\longrightarrow} |z-1|^2$$

Thus, the series converges for z such that $|z-1|^2 < 1 \Leftrightarrow |z-1| < 1$. So the series converges in a disk centered at 1 with radius $r = 1$.
Boundary points: $|z-1| = 1$
We have

$$\left|\frac{(z-1)^{2n}}{(1+\frac{1}{n})^n}\right| = \frac{|z-1|^{2n}}{(1+\frac{1}{n})^n} = \frac{1}{(1+\frac{1}{n})^n} \overset{n\to\infty}{\longrightarrow} \frac{1}{e} \neq 0$$

and so the series diverges on the boundary.

(c) Applying the root test we obtain:

$$\sqrt[n]{n^n|(z+2)^n|} = n|z+2| \overset{n\to\infty}{\longrightarrow} \infty \text{ for } z \neq -2; \quad n|z+2| = 0 \overset{n\to\infty}{\longrightarrow} 0 \text{ for } z = -2$$

Hence the series converges only at $z = -2$.

Exercise 42: For which $x \in$ does the power series

$$\sum_{n=1}^{\infty} (-2)^n \frac{n^2+2}{n^3+n} x^{3n}$$

converge?

Attention: The solution set is an interval. Does it contain its boundary points?

Solution:

For $x = 0$ the series converges . For $x \neq 0$ we have $a_n := (-2)^n \frac{n^2+2}{n^3+n} x^{3n} \neq 0$ and the ratio test yields:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-2)^{n+1} \frac{(n+1)^2+2}{(n+1)^3+(n+1)} x^{3n+3}}{(-2)^n \frac{n^2+2}{n^3+n} x^{3n}}\right| = \frac{(n+1)^2+2}{(n+1)^3+n+1} \cdot \frac{n^3+n}{n^2+2} |x|^3 \overset{n\to\infty}{\longrightarrow} 2|x|^3 \overset{!}{<} 1$$

$\Rightarrow |x|^3 < \frac{1}{2} \Rightarrow$ So the series converges for $x \in \left(-\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}\right)$, and diverges for $|x| > \frac{1}{\sqrt[3]{2}}$.

Boundary points:

- $x = \frac{1}{\sqrt[3]{2}}$. We have $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n^2+2}{n^3+n}$ converges if $\frac{n^2+2}{n^3+n}$ monotone by Leibniz. Indeed the sequence $\left|\frac{n^2+2}{n^3+n}\right| \leq \frac{n^2+n^2}{n^3} = \frac{2}{n}$, tends to 0 since it is majorized by $2/n$. Let $X = [(n+1)^3+n+1]\cdot[n^3+n]$. $\frac{(n+1)^2+2}{(n+1)^3+n+1} - \frac{n^2+2}{n^3+n} = \frac{(n^3+n)(n^2+2n+3)-(n^2+2)(n^3+3n^2+4n+2)}{X} = \frac{-n^4-2n^3-6n^2-5n-4}{X} < 0$, for all n, so the sequence is monotone.
- $x = -\frac{1}{\sqrt[3]{2}}$. The series $\sum\limits_{n=1}^{\infty} \frac{n^2+2}{n^3+n} \geq \sum\limits_{n=1}^{\infty} \frac{n^2+1}{n^3+n} \geq \sum\limits_{n=1}^{\infty} \frac{(n^2+1)}{n(n^3+1)} = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ which diverges.

So the series converges for $x \in \left(-\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}\right]$.

Exercise 43: Expand the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$, $f(x) = \dfrac{e^x}{1-x}$ as a power series $(\sum_{n=0}^{\infty} a_n\, x^n)$ centered at 0.

(a) Show that $a_n = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}$ for all $n \in \mathbb{N}_0$.

(b) Find all $x \in \mathbb{R}$ where the power series converges.

Solution:

(a)
- We will use the power series for e^x to obtain:

$$(1-x) \sum_{n=0}^{\infty} a_n\, x^n \stackrel{!}{=} e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

  Using the absolute convergence of the series for e^x we have

$$(1-x) \sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=0}^{\infty} a_n \left(x^n - x^{n+1}\right) \stackrel{(*)}{=} \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = a_0 + \sum_{n=1}^{\infty} (a_n - a_{n-1})\, x^n.$$

  Setting the corresponding coefficients of x^n equal we obtain

$$a_0 = 1, \qquad a_n = a_{n-1} + \frac{1}{n!} \quad \text{for } n \geq 1.$$

  We will now prove the coefficients have the correct form by induction. We have $a_0 = 1$. Suppose that we have

$$a_n = \sum_{k=0}^{n} \frac{1}{k!}.$$

  Using the relation

$$a_{n+1} = a_n + \frac{1}{(n+1)!}$$

  we obtain

$$a_{n+1} = \sum_{k=0}^{n} \frac{1}{k!} + \frac{1}{(n+1)!} = \sum_{k=0}^{n+1} \frac{1}{k!},$$

  as required.
- Alternatively we can use the Cauchy product formula:

$$\begin{aligned} f(x) &= \frac{1}{1-x} e^x \stackrel{|x|<1}{=} \sum_{j=0}^{\infty} x^j \cdot \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} x^k \cdot \frac{1}{(m-k)!} x^{m-k} \right) \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \frac{1}{(m-k)!} x^m \right) = \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \frac{1}{k!} x^m \right) = \sum_{m=0}^{\infty} \left(x^m \left(\sum_{k=0}^{m} \frac{1}{k!} \right) \right). \end{aligned}$$

(b) We apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| x \left(\sum_{k=0}^{n+1} \frac{1}{k!} \right) \Big/ \left(\sum_{k=0}^{n} \frac{1}{k!} \right) \right| = \left| \frac{ex}{e} \right| = |x|.$$

It follows that the radius of convergence is one.
Consider the boundary point $x = 1$. We obtain $\sum_{n=0}^{\infty} a_n$. The sequence $(a_n)_n$ does not tend to 0 and so the series diverges for $x = 1$. The argument for $x = -1$ is similar. Thus the series converges on the interval $(-1, 1)$.

Exercise 44: Consider the functions $\dfrac{1}{z+1}$, $z \in \mathbb{C} \setminus \{-1\}$, and $\dfrac{1}{z+2}$, $z \in \mathbb{C} \setminus \{-2\}$.

(a) Expand both functions as power series centered at $z_0 = 1$ and find the radii of convergence.

(b) Expand the function $\left(\frac{1}{z+1}\right) \cdot \left(\frac{1}{z+2}\right)$ as a power series centered at $z_0 = 1$ using the Cauchy product. Find a closed form representation of the coefficients!

(c) Expand the function $\left(\frac{1}{z+1}\right)\cdot\left(\frac{1}{z+2}\right)$ as a power series centered at $z_0 = 1$ using partial fraction decomposition.

Hint: This means find numbers a, b such that $\left(\frac{1}{z+1}\right)\cdot\left(\frac{1}{z+2}\right) = \left(\frac{a}{z+1}\right) + \left(\frac{b}{z+2}\right)$ for all z and proceed.

(d) Find all $z \in \mathbb{C}$ where the Cauchy product from b) converges.

Solution:

(a)For a geometric series we have

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q} \text{ for } |q| < 1.$$

Since $\frac{1}{z+1}$ and $\frac{1}{z+2}$ can be written as geometric series, we have

$$\frac{1}{1+z} = \sum_{k=0}^{\infty} (-1)^k z^k q = -z) \tag{1}$$

and

$$\frac{1}{z+2} = \frac{1}{2}\frac{1}{1+\frac{z}{2}} = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k \left(\frac{z}{2}\right)^k q = -\frac{z}{2}) = \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{k+1}} z^k. \tag{2}$$

We apply the root test for (1):

$$\lim_{k\to\infty} \sqrt[k]{\left|(-1)^k z^k\right|} = |z| < 1 \Rightarrow r_1 = 1$$

We also apply the root test for (2):

$$\lim_{k\to\infty} \sqrt[k]{\left|(-1)^k z^k\right| \frac{1}{2^{k+1}}} = \frac{|z|}{2} < 1 \Leftrightarrow |z| < 2 \Rightarrow r_2 = 2$$

(b) We have

$$\begin{aligned}
\frac{1}{z+2}\cdot\frac{1}{z+1} &= \frac{1}{2}\sum_{k=0}^{\infty}(-\frac{z}{2})^k \cdot \sum_{k=0}^{\infty}(-z)^k = \frac{1}{2}\sum_{n=0}^{\infty} z^n \sum_{l=0}^{n}\left(-\frac{1}{2}\right)^l z^l(-1)^{n-l} z^{n-l} \\
&= \frac{1}{2}\sum_{n=0}^{\infty} z^n \sum_{l=0}^{n}\left(-\frac{1}{2}\right)^l (-1)^{n-l} = \frac{1}{2}\sum_{n=0}^{\infty} z^n (-1)^n \sum_{l=0}^{n}\left(\frac{1}{2}\right)^l \\
&= \frac{1}{2}\sum_{n=0}^{\infty} z^n (-1)^n \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = \frac{1}{2}\sum_{n=0}^{\infty} z^n (-1)^n \left(2 - \frac{1}{2^n}\right) \\
&= \sum_{n=0}^{\infty}(-1)^n\left(1-\frac{1}{2^{n+1}}\right) z^n.
\end{aligned} \tag{3}$$

(c) Since the difference of two power series converges in a radius equal to the minimum of the radii of convergence of those power series, we have that the series converges in a radius of 1. The root criteria can be used to easily show $r = 1$.

Exercise 45:

(a) Give the exponential representation $z = re^{i\varphi}$, $\varphi \in (-\pi, \pi]$, of the following complex numbers

$$\text{(i)} \quad -1+i, \qquad \text{(ii)} \quad (1-i)e^{-i\pi}, \qquad \text{(iii)} \quad \frac{3+4i}{1-i}, \qquad \text{(iv)} \quad \frac{1-i}{i+2}.$$

(b) Show that the following identities apply for all $z \in \mathbb{C}$:

$$\text{(i)} \quad \cos(iz) = \cosh z \text{ and } \cos z = \cosh(iz), \qquad \text{(ii)} \cos(\overline{z}) = \overline{\cos(z)}.$$

Solution:

(a) For a complex number $z = a + ib$ we have $r = |z| = \sqrt{(a^2+b^2)}$ and $\varphi = \text{Arg}(z) = \arccos \frac{a}{\sqrt{(a^2+b^2)}}$ for $b \geq 0$ and $\varphi = \text{Arg}(z) = -\arccos \frac{a}{\sqrt{(a^2+b^2)}}$ for $b < 0$.

(i) $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ and $\varphi = \arccos\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4} \quad \Rightarrow \quad -1 + i = \sqrt{2}e^{i\frac{3\pi}{4}}$

(ii) $(1-i)e^{-i\pi} = (1-i)(-1) = -1 + i \overset{(i))}{=} \sqrt{2}e^{i\frac{3\pi}{4}}$

(iii) $z = \frac{(3+4i)}{(1-i)}\frac{(1+i)}{(1+i)} = \frac{1}{2}(-1+7i)$

$r = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{7}{2}\right)^2} = \frac{5}{2}\sqrt{2}$ and $\varphi = \arccos\left(\frac{-\frac{1}{2}}{\frac{5}{2}\sqrt{2}}\right) = \arccos\left(\frac{-1}{5\sqrt{2}}\right) \approx 1,713 \quad \Rightarrow \quad z = \frac{5}{2}\sqrt{2}e^{i1,713}$.

(iv) $z = \frac{(1-i)}{(i+2)}\frac{(-i+2)}{(-i+2)} = \frac{1}{5}(1-3i)$

$r = \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{-3}{5}\right)^2} = \sqrt{\frac{2}{5}}$ and $\varphi = -\arccos\left(\frac{\frac{1}{5}}{\sqrt{\frac{2}{5}}}\right) = -\arccos\left(\frac{1}{\sqrt{10}}\right) \approx -1,249 \quad \Rightarrow \quad z = \frac{1}{\sqrt{10}}e^{-i1,249}$.

(b) (i)

$$\cos iz = (e^{iiz} + e^{-iiz})/2 = (e^{-z} + e^{z})/2 = \cosh z.$$

We have

$$\cosh iz = \cos(iiz) = \cos(-z) = \cos z$$

(ii) For $z = x + iy$ we obtain:

$$e^{i\overline{z}} = e^{i\overline{(x+iy)}} = e^{i\overline{x-y}} = e^{-ix-y} = e^{-y}e^{-ix} = e^{-y}\left(\cos(-x) + i\sin(-x)\right) = e^{-y}\left(\cos(x) - i\sin(x)\right)$$
$$\overline{e^{-y}\left(\cos(x) + i\sin(x)\right)} = \overline{e^{-y+ix}} = \overline{e^{i(iy+x)}} = \overline{e^{iz}}$$

So, $e^{i\overline{z}} = \overline{e^{iz}}$. Analogously $e^{\overline{-iz}} = \overline{e^{-iz}}$. Thus:

$$2\cos\overline{z} = e^{i\overline{z}} + e^{-i\overline{z}} = e^{\overline{-iz}} + e^{\overline{iz}} = \overline{e^{-iz}} + \overline{e^{iz}} = \overline{e^{-iz} + e^{iz}} = 2\overline{\cos z}$$

So, $\cos\overline{z} = \overline{\cos z}$.

Due date: Your written solutions are due at 14:00 on Tuesday, **8 January, 2018**.
Please submit them at the beginning of the problem session.

Website: For detailed information regarding this course visit the following web page:

`http://www.math.kit.edu/iag6/edu/am12018w/en`