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Karlsruhe, November 8, 2018

Name: ....

Exercise Sheet No. 4
Advanced Mathematics I

Exercise 16: Let the sequence $(x_n)_n$ be recursively defined by

$$x_1 = 0, \quad x_{n+1} = \frac{1}{2}(1 - x_n^2), \quad n \in \mathbb{N}.$$

(a) Compute x_j explicitly for $j \in \{1, 2, 3, 4\}$ and show by mathematical induction that the following estimation is true for all $n \in \mathbb{N}$:

$$0 \leq x_n \leq \frac{1}{2}.$$

(b) Conclude that the inequality $|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|$ holds for all $n \in \mathbb{N}$, $n \geq 2$, and prove by mathematical induction that

$$|x_{n+1} - x_n| \leq \left(\frac{1}{2}\right)^n, \quad n \in \mathbb{N}.$$

Solution:

a) $x_1 = 0$, $x_2 = 1/2$, $x_3 = 3/8$, $x_4 = 55/128$.

Base Case $n = 1 : x_1 = 0$, so $0 \leq x_1 \leq 1/2$.

Induction Step $n \to n + 1$: Assume $0 \leq x_n \leq 1/2$ holds for smaller $n \in \mathbb{N}$. We have $x_{n+1} = \frac{1}{2}(1 - x_n^2) \leq \frac{1}{2}(1 - 0) = \frac{1}{2}$ so $x_{n+1} \geq \frac{1}{2}(1 - (\frac{1}{2})^2) = 3/8 \geq 0$.

b) $|x_{n+1} - x_n| = |\frac{1}{2}(1 - x_n^2) - \frac{1}{2}(1 - x_{n-1}^2)| = |\frac{1}{2}(x_{n-1}^2 - x_n^2)| = \frac{1}{2}|x_{n-1} - x_n|(x_{n-1} + x_n) \leq \frac{1}{2}|x_{n-1} - x_n|(\frac{1}{2} + \frac{1}{2})$

Base Case $n = 1 : |x_2 - x_1| = |1/2 - 0| = (1/2)^1$.

Induction Step $n \to n + 1$: Assume $|x_{n+1} - x_n| \leq (1/2)^n$ holds for smaller $n \in \mathbb{N}$. We have $|x_{n+2} - x_{n+1}| \leq \frac{1}{2}|x_{n+1} - x_n| \leq \frac{1}{2}(\frac{1}{2})^n = (\frac{1}{2})^{n+1}$.

Exercise 17: Let the sequences $(a_n)_n$ and $(b_n)_n$ be defined by:

$$a_n = n^2 + 1, \qquad b_n = \frac{n^3 + n^2 + 3n + 1}{n^4 + n^2 - 3}, \quad n \in \mathbb{N}.$$

Compute the first 6 terms of each sequence $(a_n)_n$ and $(b_n)_n$. Which of the sequences are bounded? Give a formal proof of your answer.

Solution:

n	1	2	3	4	5	6
a_n	2	5	10	17	26	37
b_n	-6	$\frac{19}{17}$	$\frac{46}{87}$	$\frac{93}{269}$	$\frac{166}{647}$	$\frac{271}{1329}$

- The sequence $(a_n)_n$ is not bounded. We will prove this by contradiction. Suppose there was $r > 0$ such that $|a_n| < r$ for all $n \in \mathbb{N}$. Define

  $$n_0 = \min\{n \in \mathbb{N} \ : \ n > \sqrt{r - 1}\}.$$

  We have $a_{n_0} = n_0^2 + 1 > \sqrt{r - 1}^2 + 1 = r$, which is a contradiction.

- The sequence $(b_n)_n$ is bounded. We will show by induction that $b_n \leq 1$ for all $n \geq 3$. The statement

$$b_n = \frac{n^3+n^2+3n+1}{n^4+n^2-3} \leq 1 \text{ is equivalent to } n^3+n^2+3n+1 \leq n^4+n^2-3 \text{ for } n \geq 3.$$

We will prove the inequality by complete induction:

Base Case: For $n = 3$ we have: $27 + 9 + 9 + 1 = 46 \leq 87 = 81 + 9 - 3$

Induction Step: $n \to n+1$:

$$\begin{aligned}(n+1)^3 + (n+1)^2 + 3(n+1) + 1 &= \underbrace{n^3 + n^2 + 3n + 1}_{\text{IH}} + 3n^2 + 2n + 1 + 1 + 3n + 3.\\ &\leq n^4 + n^2 - 3 + 3n^2 + 5n + 5\\ &\leq n^4 + n^2 - 3 + 4n^3 + 6n^2 + 6n + 2\\ &= (n+1)^4 + (n+1)^2 - 3\end{aligned}$$

Exercise 18: Compute the limit of each of the following sequences

(a) $a_n = \frac{n^4-2}{n^2+4} + \frac{n^3(3-n^2)}{n^3+1}$

(b) $b_n = \left(1 + \left(-\frac{3}{5}\right)^n\right) \cdot \left(\frac{10^n}{n!} - \frac{3n^2+1}{(2n+1)^2}\right)$

(c) $c_n = \sqrt[n]{17 \cdot 2^n}\left(\sqrt{n+1} - \sqrt{n}\right)$.

Solution:

(a) Remark: The identity $\lim(x_n + y_n) = \lim x_n + \lim y_n$ holds only when the limits exist. We have

$$\begin{aligned}a_n &= \frac{n^4-2}{n^2+4} - \frac{n^3(n^2-3)}{n^3+1} = \frac{n^2\left(n^2+4\right) - 4n^2 - 2}{n^2+4} - \frac{\left(n^3+1\right)\left(n^2-3\right) - n^2 + 3}{n^3+1}\\ &= n^2 - \frac{4n^2+2}{n^2+4} - n^2 + 3 + \frac{n^2-3}{n^3+1} = 3 - \frac{4+\frac{2}{n^2}}{1+\frac{4}{n^2}} + \frac{\frac{1}{n} - \frac{3}{n^3}}{1+\frac{1}{n^3}}.\end{aligned}$$

It follows that

$$\lim_{n\to\infty} a_n = 3 - \frac{\lim\limits_{n\to\infty}\left(4+\frac{2}{n^2}\right)}{\lim\limits_{n\to\infty}\left(1+\frac{4}{n^2}\right)} + \frac{\lim\limits_{n\to\infty}\frac{1}{n} - \lim\limits_{n\to\infty}\frac{3}{n^3}}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n^3}\right)} = 3 - \frac{4}{1} + \frac{0}{1} = -1.$$

Alternatively we may do the following:

$$\begin{aligned}a_n &= \frac{n^4-2}{n^2+4} - \frac{n^3(n^2-3)}{n^3+1} = \frac{(n^4-2)(n^3+1) - (n^2+4)n^3(n^2-3)}{(n^2+4)(n^3+1)}\\ &= \frac{-n^5+n^4+10n^3-2}{n^5+4n^3+n^2+4} = \frac{-1+\frac{1}{n}+\frac{10}{n^2}-\frac{2}{n^5}}{1+\frac{4}{n^2}+\frac{1}{n^3}+\frac{4}{n^5}} \xrightarrow{n\to\infty} \frac{-1}{1} = -1.\end{aligned}$$

(b) We will consider the factors seperately - for the first factor observe $|-\frac{3}{5}| < 1$ and so $\lim_{n\to\infty}\left(-\frac{3}{5}\right)^n = 0$ (as a geometric sequence). In the second factor, we have $x_n := \frac{10^n}{n!} > 0$. We will show that (x_n) tends to 0 by majorizing it with a sequence (y_n) tending to 0 (i.e. applying the squeeze theorem with the 0-sequence, and a new sequence (y_n)). Let us define $(y_n)_n$, $n \in \mathbb{N}_{\geq 10}$ so that $y_n = \frac{10^n}{10!11^{n-10}}$. We have

$$\lim_{n\to\infty}\frac{10^n}{10!11^{n-10}} = \lim_{n\to\infty}\frac{10^{10}}{10!}\frac{10^{n-10}}{11^{n-10}} = \frac{10^{10}}{10!}\cdot\frac{10^{-10}}{11^{-10}}\cdot\lim_{n\to\infty}\left(\frac{10}{11}\right)^n = 0\,,$$

since $\frac{10}{11} < 1$. For $n > 10$ the sequence $(y_n)_n$ majorizes $(x_n)_n$ hence

$$\lim_{n\to\infty}\frac{10^n}{n!} = \lim_{n\to\infty}\left|\frac{10^n}{n!} - 0\right| \leq \lim_{n\to\infty}|y_n| = \lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(\frac{10^{10}}{10!}\cdot\frac{10^{n-10}}{11^{n-10}}\right) = \frac{10^{10}}{10!}\cdot\frac{10^{-10}}{11^{-10}}\cdot\lim_{n\to\infty}\left(\frac{10}{11}\right)^n = 0.$$

We now rearrange the second term in the second factor:

$$\frac{3n^2+1}{(2n+1)^2} = \frac{3n^2+1}{4n^2+4n+1} = \frac{3+\frac{1}{n^2}}{4+\frac{1}{n}+\frac{1}{n^2}}.$$

The limit is thus $\frac{3}{4}$. It follows that

$$\lim_{n\to\infty} b_n = (1+0)\cdot\left(0 - \frac{3}{4}\right) = -\frac{3}{4}.$$

(c) Multiplying and dividing by $\sqrt{n+1}+\sqrt{n}$, we have:

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} 2\sqrt[n]{17}\left(\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right) = 2\cdot 1\cdot 0 = 0.$$

Exercise 19: Calculate the limits of the complex sequences

$$\text{(a) } a_n = 2 + \frac{3}{4i\,n} + \left(\frac{1}{2}+\frac{1}{3}i\right)^n, \quad n\in\mathbb{N}, \qquad \text{(b) } b_n = \frac{(3i\,n+1)(2n+i)}{\sum\limits_{k=1}^{n} ik}, \quad n\in\mathbb{N}.$$

Solution: (a) We have $\frac{3}{4in} \to 0$ and also $(\frac{1}{2}+\frac{1}{3}i)^n \to 0$, since $|\frac{1}{2}+\frac{1}{3}i| < 1$. Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(2+\frac{3}{4in}+(\frac{1}{2}+\frac{1}{3}i)^n\right) = 2+0+0 = 2$$

(b) We have $\sum\limits_{k=1}^{n} ik = i\frac{n(n+1)}{2}$. Thus,

$$\begin{aligned}\lim_{n\to\infty}\frac{(3in+1)(2n+i)}{\sum\limits_{k=1}^{n} ik} &= \lim_{n\to\infty}\frac{2}{i}\cdot\frac{(3in+1)(2n+i)}{n(n+1)} = \lim_{n\to\infty}\frac{2}{i}\cdot\frac{6in^2-n+i}{n^2+n}\\ &= \lim_{n\to\infty}\frac{2}{i}\cdot\frac{6i-\frac{1}{n}+\frac{i}{n^2}}{1+\frac{1}{n}} = 12\,.\end{aligned}$$

Exercise 20: Consider the sequence $(a_n)_n$ with $a_n = \frac{n-1}{n+1}$, $n\in\mathbb{N}$. Find an index N such that $|a_n-1|<\varepsilon$ for every $n\geq N$, when

(a) $\varepsilon = \dfrac{1}{10}$ (b) $\varepsilon = \dfrac{1}{1000}$, (c) $\varepsilon > 0$ is arbitrary.

(d) Does the sequence $(a_n)_n$ converge? If so, what is the limit?

Solution:

a) We have

$$|a_n-1| = |\frac{n-1}{n+1}-1| = |\frac{n-1-(n+1)}{n+1}| = |\frac{-2}{n+1}| = \frac{2}{n+1}$$

which is less than $\frac{1}{10}$, when

$$\frac{2}{n+1} < \frac{1}{10} \text{ bzw. } 20\leq n+1 \text{ bzw. } 19\leq n.$$

And so $N\geq 19$, thus a) holds for $n\geq N$.

b) We need $\frac{2}{n+1}\leq\frac{1}{1000}$ so

$$2000\leq n+1 \text{ bzw. } 1999\leq n.$$

Thus for $N=1999$ we have the required bound.

c) We need

$$\frac{2}{n+1}\leq\varepsilon \text{ für alle } n\geq N$$

So $\frac{2}{N+1}\leq\varepsilon$ (equivalently $N\geq\frac{2}{\varepsilon}-1$.)

d) Yes, the sequence is convergent to 1.

Due date: Your written solutions are due at 14:00 on Tuesday, **20 November, 2018**.
Please submit them at the beginning of the problem session
or in the box in J101 (note the box will be emptied before the problem session).

Website: For detailed information regarding this course visit the following web page:

`http://www.math.kit.edu/iag6/edu/am12018w/en`