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Karlsruhe, November 16, 2018

Name: ....

Exercise Sheet No. 5
Advanced Mathematics I

Exercise 21: Determine all accumulation points of the following sequences

$$\text{(a)}\quad a_n := \frac{1}{n} + 2(-1)^n, \quad n \in \mathbb{N}, \qquad\qquad \text{(b)}\quad b_n := \left(\frac{5n+7}{n}\right) i^n, \quad n \in \mathbb{N}.$$

Solution: (a) Consider the following subsequences of $(a_n)_{n\in\mathbb{N}}$, $(a_n)_{n\in\mathbb{N},n \text{ even}}$ and $(a_n)_{n\in\mathbb{N},n \text{ odd}}$, consisting of the terms with even and odd index, respectively. The terms in the seqence $(a_n)_{n\in\mathbb{N},n \text{ even}}$ are of the form $\frac{1}{n}+2$ and those in $(a_n)_{n\in\mathbb{N},n \text{ odd}}$ are of the form $\frac{1}{n}-2$

We will make use the "accumulation point theorem" from the next. Namely, if a sequence converges then its subsequences converge to the same limit. Since $(1/n)_{n\in\mathbb{N},n \text{ even}}$ and $(1/n)_{n\in\mathbb{N},n \text{ odd}}$ are subsequences of $(1/n)_{n\in\mathbb{N}}$, and $(1/n)_{n\in\mathbb{N}}$ converges to 0, then so too must $(1/n)_{n\in\mathbb{N},n \text{ even}}$ and $(1/n)_{n\in\mathbb{N},n \text{ odd}}$.

Since we know the sequence $(1/n)_{n\in\mathbb{N}}$ converges to 0, and $(\frac{1}{2k})_{k\in\mathbb{N}}$ and $(\frac{1}{2k-1})_{k\in\mathbb{N}}$ are subsequences of this sequence, it follows they also converge to 0. Since the sum of two convergent sequences converges to a the sum of the limits we obtain that $(a_n)_{n\in\mathbb{N},n \text{ even}}$ converges to 2 and $(a_n)_{n\in\mathbb{N},n \text{ odd}}$ converges to -2.

Now we argue that there are no additional accumulation points. To do this, let L be an arbitrary accumulation point, and we will show $L \in \{-2, 2\}$. By definition there is a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with limit L. This subsequence has either infinitely many terms with odd index or infinitely many terms with even index. Suppose we have infinitely many terms with even index. The sequence $(a_{n_k})_{k\in\mathbb{N},n_k \text{ even}}$ is a subsequence of both $(a_{n_k})_{k\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N},n \text{ even}}$, and thus the limit of $(a_{n_k})_{k\in\mathbb{N},n_k \text{ even}}$ is both L and 2. It follows that $L = 2$. Identical reasoning shows that $L = -2$ in the odd case, and the proof is complete.

(b) Since $i^{4k} = 1$ we have:

$$i^{4k} = i^0 = 1,\ i^{4k+1} = i^1 = i,\ i^{4k+2} = i^2 = -1,\ i^{4k+3} = i^3 = -i.$$

Consider the following subsequences:

$$\begin{aligned} c_n &:= \left(\frac{5n+7}{n}\right) \text{ for } n = 4k,\ k \in \mathbb{N}, \\ d_n &:= \left(\frac{5n+7}{n}\right) i \text{ for } n = 4k+1,\ k \in \mathbb{N}, \\ e_n &:= -\left(\frac{5n+7}{n}\right) \text{ for } n = 4k+2,\ k \in \mathbb{N}, \\ f_n &:= -\left(\frac{5n+7}{n}\right) i \text{ for } n = 4k+3,\ k \in \mathbb{N} \end{aligned}$$

We have $c_n \to 5$, $d_n \to 5i$, $e_n \to -5$ and $f_n \to -5i$ for $n \to \infty$. Applying the accumulation point theorem in a similar manner as in part (a) shows that every subsequence converges to one of these four points.

Exercise 22: Consider the recursively defined sequence

$$a_1 = b\,, \qquad a_{k+1} = \frac{|a_k|}{2a_k - 1}\,,\ k \in \mathbb{N}$$

for two initial values $b = -\frac{1}{4}$, and $b = \frac{1}{4}$.

(a) Assume that for fixed b the sequence (a_k) converges. What are the candidates for the limit?

(b) Determine for which initial value $b = -\frac{1}{4}$ or $b = \frac{1}{4}$ the sequence is monotone.

(c) Determine for which initial value the sequence is bounded.

(d) Justify, for both initial values $b = -\frac{1}{4}$ or $b = \frac{1}{4}$, whether the sequence is converging or not. In case of convergence determine the limit.

Solution:

(a) We are assuming (a_k) converges, so let a be the limit.

$$\lim_{k\to\infty} a_{k+1} = \lim_{k\to\infty} a_k = a = \lim_{k\to\infty} \frac{|a_k|}{2a_k - 1} = \frac{|a|}{2a-1}$$

We obtain the equation $2a^2 - a = |a|$ which we break into two cases.

For $a \le 0$, we have

$$2a^2 - a = -a \iff a = 0.$$

For $a > 0$, we have

$$2a^2 - a = a \iff 2a(a-1) = 0.$$

Thus the candidate limits are 0 and 1. Remark: For the rest of the solution we use (a_k) to denote the given sequence with initial value $-1/4$ and (b_k) to denote the given sequence with initial value $1/4$.

(b) For $a_1 = -\frac{1}{4}$ we have $a_2 = -\frac{1}{6}$, $a_3 = -\frac{1}{8}$. We have $a_k < 0$ for all $k \in \mathbb{N}$. We will show this by induction:

Base Case: $a_1 = -\frac{1}{4} < 0$.

Induction Step $k \curvearrowright k+1$: Assume we have $a_k < 0$ for all $k \in \mathbb{N}$. We show: $a_{k+1} = \frac{|a_k|}{2a_k - 1} < 0$. Indeed, the numerator is an absolute value and thus positive, the denominator is negative by induction.

We use this result to show monotonicity:

$$\begin{aligned} a_{k+1} - a_k &= \frac{|a_k|}{2a_k - 1} - a_k = \frac{-a_k - 2a_k^2 + a_k}{2a_k - 1} \\ &= -2\frac{a_k^2}{2a_k - 1} = -2|a_k|\frac{|a_k|}{2a_k - 1} > 0 \quad (*) \end{aligned}$$

Thus the sequence $(a_k)_k$ is monotone.

For the sequence $(b_k)_k$ we have $b_1 = \frac{1}{4}$, $b_2 = -\frac{1}{2}$ and $b_3 = -\frac{1}{4}$, so the sequence is not monotone.

(c) (a_k) is monotone increasing and bounded above by 0 as shown in (b). Thus (a_k) is bounded.

With the exception of b_1, the sequence (b_k) is also strictly negative. This may be shown simply again by induction as in (b). From the negativity of (b_k) for $k \ge 2$ we can again show that b_k is strictly increasing for $k \ge 2$. Thus $-1/2 \le b_k \le 1/4$ and so (b_k) is a bounded sequence.

(d) Since (a_k) is monotone and increasing it converges. Since (a_k) is strictly negative 1 cannot be the limit, thus $a = 0$ is the limit.

The sequence (b_k) is also monotone and bounded for $k \ge 2$ and a finite number (1) of initial terms does not affect the convergence. Since b_k is strictly negative for $k \ge 2$, the only possible limit is $a = 0$.

Exercise 23: Let the sequence $(c_n)_{n=1}^{\infty}$ in $\mathbb{C}$ be recursively defined by

$$c_1 = 1 + 2i \qquad \text{and} \qquad c_{n+1} = \frac{2\mathrm{Re}(c_n)\mathrm{Im}(c_n)}{\mathrm{Re}(c_n) + \mathrm{Im}(c_n)} + i\sqrt{\mathrm{Re}(c_n)\mathrm{Im}(c_n)} \quad \text{for all } n \in \mathbb{N}.$$

(a) Show that $\mathrm{Re}(c_1) \le \mathrm{Re}(c_2) \le \ldots \le \mathrm{Re}(c_n) \le \mathrm{Im}(c_n) \le \ldots \le \mathrm{Im}(c_2) \le \mathrm{Im}(c_1)$ for all $n \in \mathbb{N}$.

(b) Show that (c_n) converges and that $\mathrm{Re}(c) = \mathrm{Im}(c)$ holds for the limit c of the sequence.
Note: You don't have to compute c.

Solution:

(a) We proceed by induction:

Base Case: For $n = 1$ we have $\mathrm{Re}(c_1) = 1 \le 2 = \mathrm{Im}(c_1)$. Thus the result holds for $n = 1$.

Induction Step: For $n \in \mathbb{N}$ we have $\mathrm{Re}(c_1) \le \ldots \le \mathrm{Re}(c_n) \le \mathrm{Im}(c_n) \le \ldots \le \mathrm{Im}(c_1)$. We will show the following inequalities:

$$\mathrm{Re}(c_n) \overset{(i)}{\le} \mathrm{Re}(c_{n+1}) \overset{(ii)}{\le} \mathrm{Im}(c_{n+1}) \overset{(iii)}{\le} \mathrm{Im}(c_n)$$

For (i):

$$\mathrm{Re}(c_{n+1}) = \frac{2\mathrm{Re}(c_n)\mathrm{Im}(c_n)}{\mathrm{Re}(c_n) + \mathrm{Im}(c_n)} \overset{\mathrm{Re}(c_n)\le\mathrm{Im}(c_n)}{\ge} \frac{2\mathrm{Re}(c_n)\mathrm{Im}(c_n)}{2\mathrm{Im}(c_n)} = \mathrm{Re}(c_n).$$

For (ii): We use the identity that for positive a, b we have $(\sqrt{a} - \sqrt{b})^2 = a + b - 2\sqrt{ab} \ge 0$. By the induction hypothesis, $0 < \mathrm{Re}(c_1) \le \mathrm{Re}(c_n) \le \mathrm{Im}(c_n)$, and we obtain,

$$\mathrm{Re}(c_n) + \mathrm{Im}(c_n) \geq 2\sqrt{\mathrm{Re}(c_n)\mathrm{Im}(c_n)}, \qquad \text{so} \qquad \frac{2\mathrm{Re}(c_n)\mathrm{Im}(c_n)}{\mathrm{Re}(c_n) + \mathrm{Im}(c_n)} \leq \sqrt{\mathrm{Re}(c_n)\mathrm{Im}(c_n)}.$$

For (iii):

$$\mathrm{Im}(c_{n+1}) = \sqrt{\mathrm{Re}(c_n)\mathrm{Im}(c_n)} \overset{\mathrm{Re}(c_n)\leq\mathrm{Im}(c_n)}{\leq} \sqrt{\mathrm{Im}(c_n)\mathrm{Im}(c_n)} = \mathrm{Im}(c_n).$$

(b) In (a) it has already been shown that $(\mathrm{Re}(c_n))$ is monotone increasing and bounded. Similarly, $(\mathrm{Im}(c_n))$ is monotone decreasing and bounded. Thus, both these sequences converge and so too must (c_n).

The real and imaginary parts of the limit c must satisfy:

$$\mathrm{Im}(c) = \sqrt{\mathrm{Re}(c)\mathrm{Im}(c)}, \qquad \text{so} \qquad \sqrt{\mathrm{Im}(c)} = \sqrt{\mathrm{Re}(c)}.$$

Thus, we obtain $\mathrm{Im}(c) = \mathrm{Re}(c)$.

Exercise 24: Which of the following assertions are true? Give a counter example for each incorrect assertion.

(a) If a sequence is monotone and bounded, then it converges.

(b) If a sequence converges, then it is monotone and bounded.

(c) If a sequence is not bounded, then it is not convergent.

(d) If a sequence is not monotonic, then it is not convergent.

(e) If a sequence has exactly one accumulation point, then it converges.

(f) If a sequence converges, then it has exactly one accumulation point.

Solution:

(a) True.

(b) False. The seqeunce $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ converges to 0, but it is not monotone

(c) True.

(d) False. This statement is the contrapositive of the one in (b) and thus logically equivalent.

(e) False. The sequence a_n defined to be 1, for n odd and n, for n even has one accumulation point but diverges as it is unbounded.

(f) True.

Exercise 25:

Determine the image of the set $M := \{z \in \mathbb{C} : |z| = 1\}$ under the mapping

$$f : \begin{cases} \mathbb{C} \setminus \{-3/2 + i/2\} \longrightarrow \mathbb{C}, \\ z \mapsto \dfrac{-2}{2z + 3 - i}. \end{cases}.$$

Sketch the sets M and $f(M)$.

Solution: Define $g(z) := z + 3/2 - i/2$, $h(z) := 1/z$ and $k(z) := -z$. We have

$$f(z) = -\frac{1}{z + 3/2 - i/2} = -\frac{1}{g(z)} = -h(g(z)) = k(h(g(z))).$$

The function g is simply a translation (shift) by $3/2 - i/2$, so under this function, M is mapped to a circle with center $3/2 - i/2$ and radius 1.

It was shown in the script that the function h maps a circle of radius R and centert z_0 to a circle of radius $\frac{R}{||z_0|^2 - R^2|}$ and center $\frac{\bar{z}_0}{|z_0|^2 - R^2}$. The circle with center $3/2 - i/2$ and radius 1 is mapped under h to the circle with center $1 + i/3$ and radius $2/3$.

The function k is a rotation by π, equivalently a reflection through the origin. Thus we end with a circle with center $-1 - i/3$ and radius $2/3$.

Due date: Your written solutions are due at 14:00 on Tuesday, **27 November, 2018**.
Please submit them at the beginning of the problem session.

Website: For detailed information regarding this course visit the following web page:

http://www.math.kit.edu/iag6/edu/am12018w/en