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Karlsruhe, December 6, 2018

Name: ....

Exercise Sheet No. 8
Advanced Mathematics I

Exercise 36:

Given the series $\left(\sum\limits_{n=1}^{\infty} a_n\right)$ and $\left(\sum\limits_{n=m}^{\infty} b_n\right)$, where $a_n = \frac{1}{n(n+1)}$, $b_n = \frac{2}{3^n}$ for $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

(a) Determine the first partial sums of the series $\left(\sum\limits_{n=1}^{\infty} a_n\right)$ and $\left(\sum\limits_{n=m}^{\infty} b_n\right)$.

(b) Show that $\left(\sum\limits_{n=1}^{\infty} a_n\right)$ converges and find the limit $\sum\limits_{n=1}^{\infty} a_n$.

(c) Show that $\left(\sum\limits_{n=m}^{\infty} b_n\right)$ converges and find the limit $\sum\limits_{n=m}^{\infty} b_n$.

Solution:

(a) $\left(\sum\limits_{n=1}^{\infty} a_n\right)$:

$S_1 = \sum\limits_{n=1}^{1} \frac{1}{n(n+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$

$S_2 = \sum\limits_{n=1}^{2} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{2(2+1)} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

$S_3 = \sum\limits_{n=1}^{3} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{3(3+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$

$S_4 = \sum\limits_{n=1}^{4} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{4(4+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}$

$S_5 = \sum\limits_{n=1}^{5} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{5(5+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{5}{6}$

$\left(\sum\limits_{n=m}^{\infty} b_n\right)$:

$S_m = \sum\limits_{n=m}^{m} \frac{2}{3^n} = \frac{2}{3^m}$

$S_{m+1} = \sum\limits_{n=m}^{m+1} \frac{2}{3^n} = \frac{2}{3^m} + \frac{2}{3^{m+1}} = \frac{8}{3^{m+1}}$

$S_{m+2} = \sum\limits_{n=m}^{m+2} \frac{2}{3^n} = \frac{2}{3^m} + \frac{2}{3^{m+1}} + \frac{2}{3^{m+2}} = \frac{26}{3^{m+2}}$

$S_{m+3} = \sum\limits_{n=m}^{m+3} \frac{2}{3^n} = \frac{2}{3^m} + \frac{2}{3^{m+1}} + \frac{2}{3^{m+2}} + \frac{2}{3^{m+3}} = \frac{80}{3^{m+3}}$

$S_{m+4} = \sum\limits_{n=m}^{m+4} \frac{2}{3^n} = \frac{2}{3^m} + \frac{2}{3^{m+1}} + \frac{2}{3^{m+2}} + \frac{2}{3^{m+3}} + \frac{2}{3^{m+4}} = \frac{242}{3^{m+4}}$

(b) We want an expression in the form:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}.$$

To solve for A and B multiply through by $n(n+1)$ to obtain the equation

$$1 = A(n+1) + Bn,$$

equivalently,

$$1 = A + n(A+B).$$

This yields

$$A = 1, A + B = 0.$$

So, $A = 1, B = -1$:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Now we take advantage of the telescoping sum:

$$\begin{aligned}
S_n &= \sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)\\
&= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\\
&= 1+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\ldots+\left(-\frac{1}{n}+\frac{1}{n}\right)-\frac{1}{n+1}\\
&= 1-\frac{1}{n+1} \overset{n\to\infty}{\longrightarrow} 1.
\end{aligned}$$

Thus (S_n) converges to 1 and so the series $\sum_{n=1}^{\infty} a_n$ equals 1.

(c) We find the general form for a partial sum, taking care with the indices:

$$S_n = \sum_{k=m}^{n}\frac{2}{3^k} = \sum_{k=0}^{n-m}\frac{2}{3^{k+m}} = \frac{2}{3^m}\sum_{k=0}^{n-m}\frac{1}{3^k} = \frac{2}{3^m}\frac{1-(1/3)^{n-m+1}}{1-1/3}$$

Finally we take a limit to obtain the value of the series:

$$S_n \overset{n\to\infty}{\longrightarrow} \frac{2}{3^m}\cdot\frac{1}{1-\frac{1}{3}} = \frac{2}{3^m}\cdot\frac{3}{2} = \frac{1}{3^{m-1}}.$$

Exercise 37: Determine if the following series are convergent or absolutely convergent

$$\text{(a)}\ \left(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k}{2^k}\right), \qquad \text{(b)}\ \left(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sqrt{k+\frac{1}{k}}}{k}\right)$$

In the case of convergence find an index N after which the partial sums s_n for $n \geq N$ differ by no more than 10^{-2} from the limit.

Solution: Notation: $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$.

(a) For all $k \in \mathbb{N}$, $a_k = \frac{k}{2^k} > 0$. We have

$$\frac{a_{k+1}}{a_k} = \frac{k+1}{2^{k+1}}\frac{2^k}{k} = (1+\frac{1}{k})\frac{1}{2} \leq 1, \qquad k \in \mathbb{N}.$$

So (a_k) is monotone decreasing. Clearly $\lim_{k\to\infty} a_k = 0$, and so by Leibniz criterium we have $\left(\sum(-1)^{k+1}a_k\right)$ converges.

The ratio test can be used to show absolute convergence:

$$\left|\frac{(-1)^{k+2}\ a_{k+1}}{(-1)^{k+1}\ a_k}\right| = \frac{a_{k+1}}{a_k} = (1+\frac{1}{k})\frac{1}{2} \longrightarrow \frac{1}{2} < 1 \quad \text{for } k\to\infty.$$

Error estimate: The Leibniz criterium provides an error estimate: $|s-s_n| \leq a_{n+1}$ for the partial sums s_n and limit s. We want n, for which $a_{n+1} \leq 10^{-2}$ ist. By the binomial theorem $2^n = (1+1)^n \geq n(n-1)/2$, $n \in \mathbb{N}$, so

$$a_{n+1} = \frac{n+1}{2^{n+1}} = \frac{1}{2}\frac{n+1}{2^n} \leq \frac{1}{2}\frac{n+n}{n(n-1)/2} = \frac{2}{n-1}.$$

We require $\frac{2}{n-1} \leq 10^{-2}$ which happens for $n \geq N := 201$.

(b) We will again apply the Leibniz criterium. Observe

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{\sqrt{k+\frac{1}{k}}}{k} = \lim_{k\to\infty} \sqrt{\frac{k^2+1}{k^3}} = \lim_{k\to\infty} \sqrt{\frac{1}{k}+\frac{1}{k^3}} = 0.$$

Moreover, (a_k) is monotone since

$$\frac{a_{k+1}}{a_k} = \frac{\sqrt{k+1+\frac{1}{k+1}}}{k+1}\,\frac{k}{\sqrt{k+\frac{1}{k}}} = \sqrt{\frac{(k+1+\frac{1}{k+1})k^2}{(k+1)^2(k+\frac{1}{k})}}$$
$$\leq \sqrt{\frac{(k+1+1)k^2}{(k+1)^2k}} = \sqrt{\frac{(k+2)k}{(k+1)^2}} = \sqrt{\frac{k^2+2k}{k^2+2k+1}} < 1.$$

Thus, the series $\left(\sum(-1)^{k+1}a_k\right)$ converges by Leibniz.

Since $a_k \geq \frac{\sqrt{k}}{k}$ and $\left(\sum \frac{1}{\sqrt{k}}\right)$ diverges, it follows that our series is not absolutely divergent.

Error Estimate: We have $a_{n+1} = \frac{\sqrt{n+1+\frac{1}{n+1}}}{n+1} < \frac{\sqrt{n+1+n+1}}{n+1} = \frac{\sqrt{2}}{\sqrt{n+1}}$. Thus, $|s-s_n| \leq 10^{-2}$ occurse when $a_{n+1} < \frac{\sqrt{2}}{\sqrt{n+1}} \leq 10^{-2}$. Hence we require $n \geq N := 2\cdot 10^4 - 1$.

Exercise 38:

Determine whether the following sequences converge using the quotient or root criteria.

(a) $\left(\sum_{k=1}^{\infty}\left(\frac{10}{9}+\frac{1}{k}\right)^k\right)$, (b) $\left(\sum_{k=1}^{\infty}\frac{(k+1)!}{(2k)!}\right)$, (c) $\left(\sum_{k=1}^{\infty}2^{-\frac{k^2+1}{k+1}}\right)$, (d) $\left(\sum_{k=1}^{\infty}\frac{k!}{k^k}\right)$.

Solution:

(a) We apply the root test:

$$\sqrt[k]{|a_k|} = \sqrt[k]{\left(\frac{10}{9}+\frac{1}{k}\right)^k} = \frac{10}{9}+\frac{1}{k} \to \frac{10}{9}, \quad (k\to\infty).$$

Since $\frac{10}{9} > 1$, the series $\left(\sum\limits_{k=1}^{\infty} a_k\right)$ diverges.

(b) We apply the ratio test:

$$\frac{|a_{k+1}|}{|a_k|} = \frac{(k+2)!}{(2(k+1))!}\,\frac{(2k)!}{(k+1)!} = \frac{(k+1)!(k+2)(2k)!}{(2k)!(2k+2)(2k+1)(k+1)!} = \frac{k+2}{(2k+1)(2k+2)} \to 0 \quad (k\to\infty).$$

Since $0 < 1$ the series $\left(\sum\limits_{k=1}^{\infty} a_k\right)$ converges absolutely.

(c) We apply the root test:

$$\sqrt[k]{\left|2^{-\frac{k^2+1}{k+1}}\right|} = 2^{-\frac{k^2+1}{(k+1)k}} \to 2^{-1} = \frac{1}{2}, \quad (k\to\infty).$$

Since $\frac{1}{2} < 1$, the series converges absolutely.

(d) We apply the ratio test:

$$\frac{|a_{k+1}|}{|a_k|} = \frac{(k+1)!}{(k+1)^{k+1}}\,\frac{k^k}{k!} = \frac{k!(k+1)k^k}{k!(k+1)^k(k+1)} = \left(\frac{k}{k+1}\right)^k = \frac{1}{\left(1+\frac{1}{k}\right)^k} \to \frac{1}{e} \quad (k\to\infty).$$

Since $\frac{1}{e} < 1$ the series $\left(\sum\limits_{k=1}^{\infty}\frac{k!}{k^k}\right)$ converges absolutely.

Exercise 39: Show that each of the following series either converges or diverges, by constructing a larger convergent series, or smaller divergent series, respectively, and then using the comparison test.

a) $\left(\sum_{k=8}^{\infty}\frac{2k+3\sqrt{k}+3\sqrt[3]{k}}{k^3-k}\right)$, b) $\left(\sum_{k=0}^{\infty}\frac{\sqrt[4]{k+1}-\sqrt[4]{k}}{\sqrt[3]{k+2}}\right)$, c) $\left(\sum_{k=0}^{\infty}\frac{\sqrt[4]{k+1}-\sqrt[4]{k}}{\sqrt[4]{k+3}}\right)$. **Solution:**

(a) Since

$$0 \le \frac{2k + 3\sqrt{k} + 3\sqrt[3]{k}}{k^3 - k} \le \frac{2k + 6k}{k^3 - k} = \frac{8}{k^2 - 1} < \frac{8}{k^2 - \frac{1}{2}k^2} = \frac{16}{k^2}.$$

our series is majorized by a convergent series and thus converges.

(b) Since

$$\begin{aligned} 0 \le \frac{\sqrt[4]{k+1} - \sqrt[4]{k}}{\sqrt[3]{k+2}} &= \frac{\sqrt{k+1} - \sqrt{k}}{(\sqrt[4]{k+1} + \sqrt[4]{k})\sqrt[3]{k+2}} = \frac{k+1-k}{(\sqrt{k+1} + \sqrt{k})(\sqrt[4]{k+1} + \sqrt[4]{k})\sqrt[3]{k+2}} \\ &\le \frac{1}{2\sqrt{k} \cdot 2\sqrt[4]{k} \cdot \sqrt[3]{k}} = \frac{1}{4}\frac{1}{k^{\frac{1}{2}+\frac{1}{4}+\frac{1}{3}}} = \frac{1}{4}\frac{1}{k^{\frac{13}{12}}}. \end{aligned}$$

The series $\left(\sum_{k=0}^{\infty} \frac{1}{k^{\frac{13}{12}}}\right)$ converges because $\frac{13}{12} > 1$ (p-test). Thus, the given series also converges as it is majorized by a convergent series.

(c) We have

$$\begin{aligned} \frac{\sqrt[4]{k+1} - \sqrt[4]{k}}{\sqrt[4]{k+3}} &= \frac{\sqrt{k+1} - \sqrt{k}}{(\sqrt[4]{k+1} + \sqrt[4]{k})\sqrt[4]{k+3}} = \frac{k+1-k}{(\sqrt{k+1} + \sqrt{k})(\sqrt[4]{k+1} + \sqrt[4]{k}) \cdot \sqrt[4]{k+2}} \\ &\ge \frac{1}{2\sqrt{k+3} \cdot 2\sqrt[4]{k+3} \cdot \sqrt[4]{k+3}} = \frac{1}{4(k+3)}. \end{aligned}$$

Thus, the series is divergent.

Exercise 40:

(a) For which $q \in \mathbb{R}$ does the following sequence converge: $\sum_{n=0}^{\infty} (n+1)q^n$?

(b) Examine the convergence of the following sequences and determine the limit if it exists:

$$\text{(i)} \left(\sum_{n=0}^{\infty} \left(\frac{3+4i}{6}\right)^n\right), \qquad \text{(ii)} \left(\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{2}}\right).$$

Solution:

(a) Consider the root test: $\sqrt[n]{|(n+1)q^n|} = \sqrt[n]{n+1}|q| \overset{n\to\infty}{\longrightarrow} |q|$. We have absolute converge for $|q| < 1$, and divergence for $|q| > 1$. For $|q| = 1$, the series also diverges since the sequence of a_n's does not tend to 0.

(b) (i) By the geometric series formula: $\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$, $n \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{C} \setminus \{1\}$, we have

$$s_n = \sum_{k=0}^{n} \left(\frac{2}{2+3i}\right)^k = \left[1 - \left(\frac{2}{2+3i}\right)^{n+1}\right] \left[1 - \frac{2}{2+3i}\right]^{-1} = \left[1 - \left(\frac{2}{2+3i}\right)^{n+1}\right] \cdot \frac{2+3i}{3i}.$$

For $q = \frac{2}{2+3i}$ we have $|q| = 2/\sqrt{4+9} = 2/\sqrt{13} < 1$, so $q^n \to 0$ and hence

$$\sum_{k=0}^{\infty} \left(\frac{2}{2+3i}\right)^k = \frac{1}{1-q} = \frac{2+3i}{3i} = 1 - \frac{2}{3}i.$$

(ii) Since the sequence $\left(\frac{1}{\sqrt[k]{2}}\right)$ converges to 1 (rather than 0), the series must diverge.

Due date: Your written solutions are due at 14:00 on Tuesday, **18 December, 2018**. Please submit them at the beginning of the problem session.

Website: For detailed information regarding this course visit the following web page:

`http://www.math.kit.edu/iag6/edu/am12018w/en`