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Karlsruhe, Jan 17, 2019

Name: ....

Exercise Sheet No. 13
Advanced Mathematics I

Exercise 61:

Use integration by parts in order to determine the following integrals

a) $\int x \sin(2x)\, dx$, b) $\int_0^{\pi/4} \frac{x}{\cos^2(x)}\, dx$.

Solution:

(a) Let $u(x) = x$, so $u'(x) = 1$, and $v'(x) = 1/\cos^2(x)$, so $v(x) = \tan(x)$ on $(-\pi/2, \pi/2)$. We integrate by parts

$$\int_0^{\pi/4} \frac{x}{\cos^2(x)}\, dx = x \tan(x)\big|_0^{\pi/4} - \int_0^{\pi/4} \tan(x)\, dx.$$

The antiderivative of $\tan(x)$ on $(-\pi/2, \pi/2)$ is $-\ln|\cos(x)|$. It follows

$$\int_0^{\pi/4} \frac{x}{\cos^2(x)}\, dx = \frac{\pi}{4} \tan\left(\frac{\pi}{4}\right) - (-\ln|\cos(x)|)\big|_0^{\pi/4} = \frac{\pi}{4} + \ln\left(\frac{1}{\sqrt{2}}\right).$$

(b) We know $\int \ln y dy = y \ln y - y + C$, and integration by parts yields

$$\begin{aligned}\int (\ln y)^2 dy &= (y \ln y - y) \ln y - \int (y \ln y - y)\frac{1}{y} dy = y(\ln y - 1) \ln y - \int (\ln y - 1) dy \\ &= y(\ln y - 1) \ln y - [y \ln y - y - y] + C = y \ln^2 y - 2y \ln y + 2y + C \\ &= y(\ln y - 1)^2 + y + C.\end{aligned}$$

Exercise 62:

Find the antiderivatives of the functions given in (a), (b) and (c). Find the derivative of the function given in (d).

(a) $f\colon (1, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x \ln(x) \ln(\ln x)}$

(b) $g\colon (0, \infty) \to \mathbb{R}$, $g(x) = \left(\frac{\ln x}{x}\right)^2$

(c) $h\colon \mathbb{R} \to \mathbb{R}$, $h(x) = \frac{x e^{\arctan x}}{(1 + x^2)^{\frac{3}{2}}}$

(d) $k\colon \mathbb{R} \to \mathbb{R}$, $k(x) = \int_{\cos(x^3)}^{x^2} t^4 e^t dt$

Solution:

(a) We do a substitution:

$$\int \frac{dx}{x \ln x \ln(\ln x)} = \left\{\begin{matrix} u = \ln x \\ du = \frac{dx}{x} \end{matrix}\right\} = \int \frac{du}{u \ln u} = \left\{\begin{matrix} t = \ln u \\ dt = \frac{du}{u} \end{matrix}\right\} = \ln|t| + C = \ln|\ln u| + C = \ln|\ln(\ln x)| + C$$

(b) We substitute $t = \ln(x)$, that is $x = e^t$. We have

$$\begin{aligned}\int \left(\frac{\ln x}{x}\right)^2 dx &= \left\{\begin{matrix} t = \ln x \\ dt = \frac{dx}{x} \end{matrix}\right\} = \int \frac{t^2}{e^t} dt = \int t^2 e^{-t} dt \overset{\text{I.P.}}{=} -t^2 e^{-t} + 2\int t e^{-t} dt \overset{\text{I.P.}}{=} -t^2 e^{-t} + 2\left(-t e^{-t} + \int e^{-t} dt\right) \\ &= -t^2 e^{-t} - 2t e^{-t} - 2e^{-t} + C = -\frac{\ln^2 x}{x} - \frac{2 \ln x}{x} - \frac{2}{x} + C.\end{aligned}$$

We perform integration by parts with $f'(t) = e^{-t}$ and $g(t) = t^2$, so $f'(t) = e^{-t}$ and $g(t) = t$.

(c) We do a substitution $t = \arctan(x)$ so $x = \tan(t)$. We use $1 + \tan^2(t) = \frac{1}{\cos^2(t)}$, $\cos^2 t + \sin^2 t = 1$ and $\tan t = \frac{\sin t}{\cos t}$. It follows

$$\int \frac{xe^{\arctan x}}{(1+x^2)^{\frac{3}{2}}} dx = \left\{ \begin{matrix} t = \arctan x \\ dt = \frac{dx}{1+x^2} \end{matrix} \right\} = \int \frac{\tan t \cdot e^t}{\sqrt{1+\tan^2 t}} dt = \int \frac{\frac{\sin t}{\cos t} e^t}{\sqrt{\frac{1}{\cos^2 t}}} dt = \underbrace{\int \sin t \cdot e^t dt}_{=:I}$$

$$I \overset{\text{I.P.}}{=} e^t \sin t - \int e^t \cos t dt \overset{\text{I.P.}}{=} e^t \sin t - e^t \cos t - \int \sin t \cdot e^t dt = e^t \sin t - e^t \cos t - I$$

Thus, $I = \frac{1}{2}(e^t \sin t - e^t \cos t) + C$. We applied integration by parts with $f'(t) = e^t$ and $g(t) = \sin(t)$, so we have $f'(t) = e^t$ and $g(t) = \cos(t)$.

We substitute $t = \arctan(x)$ to obtain:

$$\int \frac{xe^{\arctan x}}{(1+x^2)^{\frac{3}{2}}} dx = \frac{1}{2} \left(e^{\arctan x} \sin(\arctan x) - e^{\arctan x} \cos(\arctan x) \right) + C = \frac{1}{2} \frac{(x-1)e^{\arctan(x)}}{\sqrt{1+x^2}}.$$

We have used that $\sin(x) = \frac{\tan(x)}{\sqrt{1+\tan^2(x)}}$.

(d) The Fundamental Theorem of Calculus implies that for differentiable functions u, v, w, we have

$$I = \int_{t=u(x)}^{t=v(x)} w(x) dt = w(v(x))v'(x) - w(u(x))u'(x).$$

Applying this to $u(x) = \cos(x^3)$, $v(x) = x^2$, $w(x) = x^4 e^x$, yields that

$$I = (x^2)^4 e^{x^2}(2x) - (\cos(x^3))^4 e^{\cos(x^3)}(-3x^2)(\sin(x^3))$$

Exercise 63:

Find solutions for the initial value problems:

$$\text{(a) } u' = e^{-u} \cos x, \qquad u(0) = 1, \qquad\qquad \text{(b) } uu' + (1+u^2) \sin x = 0, \qquad u(0) = 1.$$

Solution:

The differential equations are separable.

(a) Let $u'(x) = g(x)h(u(x))$ where $g(x) = \cos(x)$ and $h(z) = e^{-z}$. We calculute:

$$H(z) = \int \frac{1}{h(z)} dz = \int e^z dz = e^z \text{ and } \int_0^x \cos(t) dt = \sin(x).$$

Since $e^{u(x)} = \sin(x) + C$ we deduce

$$u(x) = \ln(C + \sin(x)).$$

Plugging in $u(0) = 1$ we find $C = e$.

(b) We have $u'(x) = -\frac{1+u^2}{u} \sin(x)$. Let $u'(x) = g(x)h(u(x))$ where $g(x) = -\sin(x)$ and $h(z) = \frac{1+z^2}{z}$. We calculate

$$H(z) = \int \frac{1}{h(z)} dz = \int \frac{z}{1+z^2} dz = \frac{1}{2} \ln(1+z^2) \text{ and } \int_0^x (-\sin(t)) dt = \cos(x) - 1.$$

We obtain $\frac{1}{2} \ln(1 + u(x)^2) = \cos(x) - 1 + c$ and it follows that

$$u(x) = \sqrt{e^{2\cos(x) - 2 + 2C} - 1}.$$

Pluggin in $u(0) = 1$, we get $1 = \sqrt{e^{2\cos(0)-2+2C} - 1} = \sqrt{e^{2C} - 1}$, so $e^{2C} = 2$. Thus, $C = \ln(\sqrt{2})$ and $u(x) = \sqrt{2e^{2\cos(x)-2} - 1}$.

Exercise 64:

Solve the following differential equations:

(a) $y'(x) - e^{-x} + y(x) - xy'(x) = xy(x)$, $\qquad y(0) = 1$

(b) $x - y^2(x) + 2x\, y(x)\, y'(x) = 0$ for $x > 0$, $\qquad y(1) = 1$. $\qquad$ *Hint:* Substitute $z(x) = y^2(x)$.

Solution:

(a) Our equation is first order linear. We rewrite the equation as

$$y' + y = \frac{e^{-x}}{1-x}.$$

With this form, the integrating factor we obtain is $\mu = e^{\int 1 dx} = e^x$. We obtain,

$$(y\mu)' = \mu \frac{e^{-x}}{1-x} = \frac{1}{1-x}.$$

Integrating yields,

$$y\mu = -\ln(1-x) + C$$

or equivalently,

$$y = e^{-x}(-\ln(1-x) + C).$$

Plugging in $y(0) = 1$, we obtain that $C = 1$.

(b) We substitute $z^2 = y$ so $y' = \frac{z'}{2y}$. We have

$$x - z(x) + xz'(x) = 0 \text{ that is } z'(x) = \frac{1}{x}z(x) - 1.$$

This is a first order linear differential equation. We put it in the form:

$$z' + \frac{-1}{x}z = -1,$$

and our integrating factor is $\mu = e^{\int \frac{-1}{x} dx} = 1/x$. We have

$$(z\mu)' = -1/x,$$

so

$$z = -\ln(x)x + Cx.$$

Putting in $z = y^2$ gives

$$y^2 = -\ln(x)x + Cx.$$

Pluggin in $y(1) = 0$ we obtain $C = 0$.

Exercise 65: Solve the initial value problem

$$y^3(x) - x^2 + xy^2(x)y'(x) = 0, \qquad y(1) = 1.$$

Solution:

We have a Bernoulli equation with $\alpha = -2$. We substitute $u(x) = (y(x))^3$ to obtain

$$u(x) - x^2 + \frac{x}{3}u'(x) = 0, \qquad u(1) = 1.$$

Putting it in the standard form,

$$u' + \frac{3}{x}u = 3x, \qquad u(1) = 1.$$

We get the integrating factor $\mu = e^{\int \frac{3}{x} dx} = x^3$. We have

$$(u\mu)' = 3x^4.$$

Integrating we obtain

$$u\mu = \frac{3}{5}x^5 + C,$$

or

$$u = \frac{3}{5}x^2 + \frac{C}{x^3}.$$

We put in $u = y^3$:

$$y^3 = \frac{3}{5}x^2 + \frac{C}{x^3}.$$

The initial condition yields $C = 2/5$ thus

$$y = (\frac{3}{5}x^2 + \frac{2}{5x^3})^{1/3}.$$

Due date: Your written solutions are due at 14:00 on Tuesday, **5 February, 2019**. Please submit them at the beginning of the problem session.

Website: For detailed information regarding this course visit the following web page:

`http://www.math.kit.edu/iag6/edu/am12018w/en`