### 1. Permutations and Combinations
- Multinomial Coefficients
- Twelvefold Way
- Cycle Decompositions

### 2. Inclusion-Exclusion-Principle & Möbius Inversion
- PIE
- Möbius Inversion Formula

### 3. Generating Functions
- Ordinary and Exponential
- Newton’s Binomial Theorem
- Recurrence Relations
4 Partitions

(Non-Crossing) Partitions of $[n]$ Ferrer Diagrams Standard Young Tableaux

5 Partially Ordered Sets

(Symmetric) Chain Partitions Dimension Posets Between Two Lines

6 Designs

Existence/Non-Existence Steiner Triple Systems etc. Latin Squares
Multinomial Coefficients

\[
\binom{n}{r_1, \ldots, r_k} = \frac{n!}{r_1! \cdots r_k!}
\]

where \( n = r_1 + \ldots + r_k \)

\# \( n \)-permutations of multiset \( M = \{r_1 \cdot t_1, \ldots, r_k \cdot t_k\} \)

Multinomial Theorem:

\[
(x_1 + \ldots + x_k)^n = \sum_{r_1 + \ldots + r_k = n} \left( \binom{n}{r_1, \ldots, r_k} x_1^{r_1} \cdots x_k^{r_k} \right)
\]
## Permutations and Combinations

<table>
<thead>
<tr>
<th>Multinomial Coefficients</th>
<th>Twelvefold Way</th>
<th>Cycle Decompositions</th>
</tr>
</thead>
</table>

### U L

- \( \binom{k}{n} \)
- \( \binom{n-1}{k-1} \)
- \( \binom{n+k-1}{k-1} \)

### L U

- 1
- \( s_{k}^{II}(n) \)
- \( \sum_{i=1}^{k} s_{i}^{II}(n) \)

### L L

- \( \binom{k}{n} n! \)
- \( s_{k}^{II}(n) k! \)
- \( k^n \)

### U U

- 1
- \( p_k(n) \)
- \( \sum_{i=1}^{k} p_i(n) \)
Permutations and Combinations

Multinomial Coefficients  Twelvefold Way  Cycle Decompositions

\[ \pi = (123)(5)(46) = 231654 = 1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 4 \rightarrow 6 \]

- unique partition into **disjoint** cycles
- exactly \( k \) cycles \( \rightarrow s_k^I(n) \) permutations
- no trivial cycles \( \rightarrow \) derangements
2 Inclusion-Exclusion-Principle & Möbius Inversion

PIE

Möbius Inversion Formula

- properties $P_1, \ldots, P_m$
- $N(S) = \{x \mid x \text{ has } P_i \text{ for all } i \in S\}$

Then

$$\sum_{S \subseteq [m]} (-1)^{|S|} |N(S)|$$

elements have none of the properties.

Usage:

Define $P_i \rightarrow$ Bound $|N(S)| \rightarrow$ Apply PIE
### Inclusion-Exclusion-Principle & Möbius Inversion

<table>
<thead>
<tr>
<th>PIE</th>
<th>Möbius Inversion Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Stronger PIE</strong></td>
<td></td>
</tr>
<tr>
<td>$g(A) = \sum_{S \subseteq A} f(S)$ ⇒ $f(A) = \sum_{S \subseteq A} (-1)^{</td>
<td>A</td>
</tr>
<tr>
<td><strong>Möbius Inversion</strong></td>
<td></td>
</tr>
<tr>
<td>$g(n) = \sum_{d</td>
<td>n} f(d)$ ⇒ $f(n) = \sum_{d</td>
</tr>
<tr>
<td><strong>General Posets</strong></td>
<td></td>
</tr>
<tr>
<td>$g(y) = \sum_{x \leq y} f(x)$ ⇒ $f(y) = \sum_{x \leq y} \mu(x, y) g(x)$</td>
<td></td>
</tr>
</tbody>
</table>
Generating Functions

Ordinary and Exponential

Newton’s Binomial Theorem

Reccurrence Relations

\[ F(x) = \sum_{n \geq 0} f_n x^n \]

\[ A(x) \cdot B(x) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} a_k b_{n-k} \right) x^n \]

unlabeled objects

\[ G(x) = \sum_{n \geq 0} g_n \frac{x^n}{n!} \]

labeled objects

\[ A(x) \cdot B(x) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{x^n}{n!} \]
### Generating Functions

<table>
<thead>
<tr>
<th>Ordinary and Exponential</th>
<th>Newton’s Binomial Theorem</th>
<th>Recurrence Relations</th>
</tr>
</thead>
</table>

**Newton’s Binomial Theorem**

\[
(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \quad \quad n \in \mathbb{R} - \{0\}
\]

\[
F(x) = 1 + x \cdot F(x)^2 \quad \Rightarrow \quad F(x) = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n
\]

**Catalan Numbers:**

\[
C_n = \frac{1}{n+1} \binom{2n}{n}
\]
# Generating Functions

## Ordinary and Exponential Newton’s Binomial Theorem Recurrence Relations

<table>
<thead>
<tr>
<th>Recurrence</th>
<th>Initial Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(A)f = g$</td>
<td>$f(0) = f_0, \ldots, f(k) = f_k$</td>
</tr>
</tbody>
</table>

### homogeneous ($g = 0$):

$$p(A) = (A - r)^k \Rightarrow f(n) = c_1 r^n + \cdots + n^{k-1} c_k r^n$$

$$p(A) = p_1(A) \cdot p_2(A) \Rightarrow f(n) = f_1(n) + f_2(n)$$

### non-homogeneous ($g \neq 0$):

$f_0(n)$ gen. sol. of hom. system $\rightarrow$ $f_1(n)$ particular sol. of non-hom. system $\rightarrow$ $f = f_0 + f_1$
Partitions

(Non-Crossing) Partitions of \([n]\)

Bell Numbers

\[ B_n = \sum_{k=0}^{n} s_k(n) \]

\# ways to split \(n\) persons in groups

Non-Crossing Partitions

counted by Catalan numbers

\[ C_n = \frac{1}{n+1} \binom{2n}{n} \]
Partitions

(Non-Crossing) Partitions of $[n]$  
Ferrer Diagrams  
Standard Young Tableaux

$p(n) = \# \text{ ways to write } n \text{ as a sum}$

11 = 5 + 3 + 2 + 1

Pentagonal Numbers $\omega_k$

$p_d^{\text{even}}(n) - p_d^{\text{odd}}(n) = \begin{cases} (-1)^k & \text{if } n = \omega_k \\ 0 & \text{otherwise} \end{cases}$

Thm. $p_{\text{odd}}(n) = p_{\text{dist}}(n)$
(Non-Crossing) Partitions of $[n]$  Ferrer Diagrams

Ferrer Diagrams

Standard Young Tableaux

$$\pi = 5264237$$

$$t(\lambda) = \frac{n!}{\prod_{(i, j)} |h_{i,j}|}$$
**Partially Ordered Sets**

| (Symmetric) Chain Partitions | Dimension | Posets Between Two Lines |

- **Dilworth’s Theorem**
  - partition into $w(P)$ chains

- **Multiset Lattices**
  - symmetric chain partition
Partially Ordered Sets

(Symmetric) Chain Partitions  Dimension  Posets Between Two Lines

\[ \dim(P) = \min \{ k \mid P = L_1 \cap \cdots \cap L_k \} \]

**Thm:** \( \dim(P) \leq w(P) \)

\[ \dim(S_n) = w(S_n) = n \]

**Thm:** \( \dim(P) \leq 2 \iff \exists L \text{ ordering all } 1 \oplus 2 \)
Partially Ordered Sets

(Symmetric) Chain Partitions  Dimension  Posets Between Two Lines

interval order  no $2 \oplus 2$

segment order  $\dim(P) \leq 2$

triangle order  $\exists L$ ordering all $2 \oplus 2$

curve order  all posets
6 Designs

| Existence/Non-Existence | Steiner Triple Systems etc. | Latin Squares |

$t$-$(v, k, \lambda)$ Design: $v$ points, blocks are $k$-sets, every $t$-tuple of points in $\lambda$ blocks

**Thm:** # blocks $= \lambda \cdot \binom{v}{t} / \binom{k}{t}$

repetition of $i$-sets $= \lambda \cdot \binom{v-i}{t-i} / \binom{k-i}{t-i}$

**Thm:** # blocks $\geq v \geq (t + 1)(k - t + 1)$

2-$(7, 3, 1)$-design
Designs

Existence/Non-Existence

Steiner Triple Systems etc.

Latin Squares

Affine Planes
2-\((n^2, n, 1)\)-designs
\(n\) prime power

Steiner Triple Systems
2-\((\nu, 3, 1)\)-designs
\(\nu \in \{1, 3\}\) (mod 6)

Projective Planes
2-\((q^2 + q + 1, q + 1, 1)\)-designs
\(q\) prime power
# Designs

<table>
<thead>
<tr>
<th>Existence/Non-Existence</th>
<th>Steiner Triple Systems etc.</th>
<th>Latin Squares</th>
</tr>
</thead>
</table>

- Each row is a permutation
- Each column is a permutation

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- $n$-by-$n$ array filled with $\mathbb{Z}_n$

**Thm.** $n - 1$ MOLS of order $n$ $\iff$ affine plane of order $n^2$ $\iff$ $n = p^k$