Problem Sheet 11

Due date: **July 10, 2017, 15:30**. Discussion of solutions: July 10, 2017.

Problem 31. **6 points**

For some positve integer n the boolean lattice of dimension n is a poset B_n whose elements are all subsets of $[n]$ and $S <_{B_n} S'$ if and only if S is a proper subset of S'. Prove that for each poset P there is an integer n such that P is a subposet of B_n.

Problem 32. **6 points**

For a poset $P = (X, \leq)$ consider the families $\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \mathcal{A}_3$ and $\mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \mathcal{C}_3$ given by

$$\mathcal{A}_1 = \{A \subseteq X \mid A \text{ antichain in } P\}, \quad \mathcal{A}_2 = \{A \in \mathcal{A}_1 \mid A \text{ maximal}\}, \quad \mathcal{A}_3 = \{A \in \mathcal{A}_2 \mid A \text{ largest}\},$$
$$\mathcal{C}_1 = \{C \subseteq X \mid C \text{ chain in } P\}, \quad \mathcal{C}_2 = \{C \in \mathcal{C}_1 \mid C \text{ maximal}\}, \quad \mathcal{C}_3 = \{C \in \mathcal{C}_2 \mid C \text{ longest}\}.$$

For each of the following statements determine all ordered pairs (i, j) with $i, j \in \{1, 2, 3\}$ such that the statement holds for all posets P. Justify your answer!

(a) There is some $A \in \mathcal{A}_i$ such that for each $C \in \mathcal{C}_j$ we have $|A \cap C| = 1$.

(b) There is some $C \in \mathcal{C}_j$ such that for each $A \in \mathcal{A}_i$ we have $|A \cap C| = 1$.

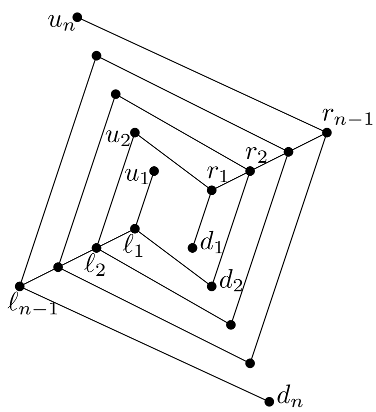
(c) For each $A \in \mathcal{A}_i$ and for each $C \in \mathcal{C}_j$ we have $|A \cap C| = 1$.

Problem 33. **6 points**

For $n \geq 2$ consider the poset $P_n = (X, \leq)$ defined as follows.

$X = \{u_1, \ldots, u_n\} \cup \{d_1, \ldots, d_n\} \cup \{\ell_1, \ldots, \ell_{n-1}\} \cup \{r_1, \ldots, r_{n-1}\}$ with relations

1) $d_i \leq \ell_j \Leftrightarrow i \geq j + 1$, 3) $\ell_i \leq u_j \Leftrightarrow i \geq j$, 5) $\ell_i \leq \ell_j \Leftrightarrow i \geq j$,
2) $d_i \leq r_j \Leftrightarrow i \leq j$, 4) $r_i \leq u_j \Leftrightarrow i \leq j - 1$, 6) $r_i \leq r_j \Leftrightarrow i \leq j$.



For each $n \geq 2$

(a) characterize all pairs (i, j) with $d_i \leq u_j$ in P_n,

(b) calculate the height of P_n,

(c) calculate the width of P_n,

(d) calculate the dimension of P_n.

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