Solution Sheet 1

Due date: **May 2, 2017, 11:00**. Discussion of solutions: May 2, 2017.

Problem 1. **6 points**

(a) Register for the homework of the lecture "Combinatorics" on the following website:

`https://iaguar.math.kit.edu/tutorienverwaltung/anmeldung.php?vorid=135`.

(b) Assume that there are n students answering part (a) and we confirm each registration individually (actually we won't!). How many different chronological orders of the total amount of $2n$ events (n registrations and n confirmations) exist? No two events occur at the same time.

Solution.

(b) Let $P(n)$ denote the number of possible chronological orders of the events for n students. Note that an answer to an e-mail is always sent later than the original registration.

We will use induction on n to prove that

$$P(n) = 2^{-n}(2n)!.$$

If $n = 1$ there is only one possible order of the 2 events and $P(1) = 1$. This gives a basis for the induction (note that we may include the case $n = 0$ also). Suppose that $n > 1$ and $P(n-1) = 2^{-(n-1)}(2(n-1))!$. The first event is certainly a registration of one of the n students, so there are n possibilities. We may send an answer to this student before or after any of the following events. Hence there are $(2(n-1)+1)P(n-1)$ possible orders of the $2n-1$ remaining events. Thus

$$P(n) = n(2n-1)P(n-1) = n(2n-1)2^{-(n-1)}(2(n-1))! = 2n(2n-1)2^{-n}(2n-2)! = 2^{-n}(2n)!$$

Note that we obtained a recurrence relation for $P(n)$ (the first equality) which can be used to find the explicit formula $2^{-n}(2n)!$. We will learn techniques for solving recurrences later. □

Problem 2. **6 points**

(a) How many distinct 4-digit numbers with digits in $\{1, 2, 3, 4, 5\}$ are there if

(i) there are no further restrictions,

(ii) all digits are distinct,

(iii) the numbers are even,

(iv) all digits are distinct and the numbers are even.

(b) There are 13 weeks of lectures in this semester. From Monday to Friday you want to solve at least one homework problem each day, but in each week you don't want to solve more than 8 exercises in total and you do not work on the weekends. Prove that there are some subsequent days (only interrupted by weekends) where you solve exactly 25 exercises in total.

Solution.

(a) (i) The set A of all 4-digit numbers with digits in $\{1, 2, 3, 4, 5\}$ can be written as

$$A = [5] \times [5] \times [5] \times [5].$$

By the multiplication principle we have

$$|A| = 5^4 = 625.$$

(ii) If the digits are distinct, then there are five choices for the first digit, four for the second, three for the third, and two for the last. Hence the total number of such numbers is

$$5 \cdot 4 \cdot 3 \cdot 2 = 120.$$

(iii) If the numbers are even, then the last digit is a 2 or a 4, the other digits are arbitrary. So the set B of such numbers can be written as

$$B = [5] \times [5] \times [5] \times \{2, 4\}.$$

By the multiplication principle we have

$$|B| = 5^3 \cdot 2 = 250.$$

(iv) Let C denote the set of even 4-digit numbers with distinct digits from $\{1, 2, 3, 4, 5\}$, let C_2 denote those numbers in C having a 2 as their last digit, and let C_4 denote those numbers in C having a 4 as their last digit. Then

$$C = C_2 \dot{\cup} C_4.$$

Let D denote the set of of 3-digit numbers with distinct digits from $\{1, 3, 4, 5\}$. Clearly, there is a bijection between the sets D and C_2. So by the bijection principle we have

$$|C_2| = |D|.$$

Similar to Part (ii) we have

$$|C_2| = |D| = 4 \cdot 3 \cdot 2 = 24.$$

With the same arguments we see that

$$|C_4| = 24.$$

By the addition principle we have

$$|C| = |C_1| + |C_2| = 24 + 24 = 48.$$

(b) There are $13 \cdot 5 = 65$ working days in your semester. For $i \in [65]$ let a_i denote the number of exercises you solved up to the i^{th} day. So a_{65} is the total number of exercises you solved during the semester. Since you solve at most 8 exercises per week we have $a_65 \leq 13 \cdot 8 = 104$. Therefore $1 \leq a_1 < \cdots < a_{65} \leq 104$. Let $b_i = a_i + 25$. Then $26 \leq b_1 < \cdots < b_{65} \leq 129$. We see that each of the numbers $a_1, \ldots, a_{65}$ as well as each of the numbers $b_1, \ldots, b_{65}$ is contained in the set $\{1, \ldots, 129\}$. Since there 130 such numbers, the pigeonhole principle yields that two of these numbers are equal. Since the numbers a_i are pairwise distinct as well as the numbers b_i, we have that $a_i = b_j = a_j + 25$ for some i, $j \in [65]$. This shows that on days $j + 1, \ldots, i$ you solved exactly 25 exercises in total. □

Problem 3. **6 points**

A sequence of numbers $a_1, \ldots, a_n$ is called *unimodal*, if there exists an $m \in \{1, \ldots, n\}$, such that $a_i \leq a_{i+1}$ for all $i < m$ and $a_i \geq a_{i+1}$ for all $i \geq m$.
Prove that the sequence $\binom{n}{1}, \ldots, \binom{n}{n}$ is unimodal for each $n \in \mathbb{N}$.

Solution.
Let $i \in \{1, \ldots, n-1\}$, $a_i = \binom{n}{i}$, and consider the fraction

$$\frac{a_{i+1}}{a_i} = \frac{n!}{(i+1)!(n-i-1)!}\frac{i!(n-i)!}{n!} = \frac{n-i}{i+1}.$$

This shows that $a_i \leq a_{i+1}$ if and only if $i \leq \frac{n-1}{2}$. In other words, the sequence $\binom{n}{1}, \ldots, \binom{n}{\lceil \frac{n}{2} \rceil}$ is increasing and the sequence $\binom{n}{\lfloor \frac{n}{2} \rfloor}, \ldots, \binom{n}{n}$ is decreasing. So the sequence $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ is unimodal (with $m = \lceil \frac{n}{2} \rceil$). □