Solution Sheet 10

Due date: **July 3, 2017, 15:30**. Discussion of solutions: July 3, 2017.

Problem 28. **6 points**

Find an explicit formula for the number of standard Young tableaux of shape $\lambda = (n, n)$, that is, with two rows of size n. Do not use the hook length formula.

Solution.

We will define a bijection between the set $SYT(n)$ of standard Young tableaux of shape $\lambda = (n, n)$ and set $W(n)$ of well formed parenthesis expressions with n pairs of parenthesis. This shows that $|SYT(n)| = C_n$ (nth Catalan number).

Consider $f : SYT(n) \to W(n)$ which maps a tableau T to the parenthesis expression w with $w(i) =' ('$ if i is in the first row of T and $w(i) =')'$ otherwise. First of all note that $f(w)$ is a well formed parenthesis expression, since if i is in column j of the tableau, then there are at least j times $'('$ and at most j times $')'$ before position i in $f(w)$ (because all numbers in the first row of the tableau up to column j are less than or equal to i).

The function is bijective, since there is an inverse function which reads the parenthesis expression w from left to right and writes number i to first empty square in the first row, if $w(i) =' ('$, and to the first empty square in the second row, if $w(i) =')'$. □

Problem 29. **6 points**

The double factorial $(2k-1)!! = \prod_{i=1}^{k}(2i-1)$ is the product over the k smallest odd natural numbers. Prove that

(a) for $k \geq 1$ we have $(2k-1)!! = \frac{(2k)!}{2^k k!}$.

(b) for $n \geq 1$ the number of standard Young tableaux with n squares is given by

$$i_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!!.$$

Solution.

(a) We have

$$\frac{(2k)!}{2^k k!} = \frac{\prod_{i=1}^{k} 2i \cdot \overbrace{\prod_{i=1}^{k}(2i-1)}^{=(2k-1)!!}}{2^k k!} = \frac{\prod_{i=1}^{k} i}{k!}(2k-1)!! = (2k-1)!!$$

(b) From the lecture it is known that the number of standard Young tableaux with n squares equals the number of involutions of $[n]$. We shall count this number. Any involution π is uniquely determined by those pairs of numbers from $[n]$ that are swapped by π. We distinguish cases based on the number k of these pairs with $0 \leq k \leq \lfloor n/2 \rfloor$.

If there are k such pairs then there are $\binom{n}{k}$ ways to choose those $2k$ numbers that shall form these pairs. We are interested in the number of ways to form unlabeled pairs from these numbers. We may calculate this as the number of permutations of a multiset with k types of repetition number 2 each, divided by $k!$ to factor out the labeled types. Hence there are $\frac{(2k)!}{2^k k!} \overset{(a)}{=} (2k-1)!!$ ways to form the pairs. Altogether we have that

$$i_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!!.$$

□

Problem 30. **6 points**

For $n \geq 0$ let s_n denote the number of permutations $\pi \in S_n$ with no increasing subsequence of length 3, i.e., no $i, j, k \in [n]$ exist with $i < j < k$ and $\pi(i) < \pi(j) < \pi(k)$.

(a) Prove that $s_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n}{k} \left(1 - \frac{k}{n-k+1} \right) \right)^2$ for all $n \geq 0$.

(b) Prove that $s_n = C_n$ for all $n \geq 0$, where C_n is the n^{th} Catalan number.

Hint: The Robinson-Schensted correspondence might help for (a). In case you need it, you may use the following identity without proof in part (b) (here $\binom{n}{-1} = 0$)

$$2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \binom{n}{k-1} = \begin{cases} \binom{2n}{n-1}, & n \text{ even}, \\ \binom{2n}{n-1} - \binom{n}{\lceil \frac{n}{2} \rceil}^2, & n \text{ odd}. \end{cases}$$

Solution.

(a) Let (T_1, T_2) denote the pair of standard Young tableaux corresponding to a permutation with no increasing subsequence of length 3. It is known from the lecture that T_1 and T_2 are of the same shape and that the first row, and hence any row, of both tableaux has length at most 2. Due to the Robinson-Schensted correspondence the number of permutations with no increasing subsequence of length 3 equals the number of pairs of standard Young tableaux of the same shape with rows of length at most 2.

Observe that a shape of n with rows of length at most 2 is determined uniquely by its number k of rows of length 2. That is for each $k \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ there is exactly one shape of n with exactly k rows of length 2. Let λ_k denote the shape of n with exactly k rows of length 2 and let f_k denote the number of standard Young tableaux of shape λ_k, i.e., the number of fillings of λ_k. Then the number of pairs of tableaux of shape λ_k equals f_k^2.

Observe that λ_k has squares (i, j) of the form $(1, j)$ for each $1 \leq j \leq n-k$ and $(2, j)$ for each $n-2k < j \leq n-k$. A hook at position (i, j) has length $\begin{cases} j+1 & \text{, if } i = 1, j > n-2k \\ j & \text{, if } i = 1, j \leq n-2k \\ j-n+2k & \text{, if } i = 2 \end{cases}$.

Due to the Hook length formula

$$f_k = \frac{n!}{\underbrace{(n-k+1)(n-k)\cdots(n-2k+2)}_{k \text{ terms, hooks,}(1,j) \text{ for } j>n-2k}\ \underbrace{(n-2k)!}_{\text{hooks,}(1,j) \text{ for } j\leq n-2k}\ \underbrace{k!}_{\text{hooks,}(2,j)}} = \binom{n}{k}\frac{n-2k+1}{n-k+1}.$$

Hence

$$s_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} f_k^2 = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n}{k}\frac{n-2k+1}{n-k+1}\right)^2 = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n}{k}(1-\frac{k}{n-k+1})\right)^2.$$

(b) **Alternative 1:** We will use the formula from part (a). Note that $\binom{n}{-1} = \binom{n}{n+1} = 0$.

$$\begin{aligned}
a_n &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n}{k}(1-\frac{k}{n-k+1})\right)^2 \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n}{k} - \binom{n}{k-1}\right)^2 \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n}{k}^2 - 2\binom{n}{k}\binom{n}{k-1} + \binom{n}{k-1}^2\right) \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k}^2 + \underbrace{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{n-k+1}^2}_{= \sum_{k=\lceil \frac{n}{2} \rceil+1}^{n} \binom{n}{k}^2} - \underbrace{2\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k}\binom{n}{k-1}}_{\text{hint applies here}} \\
&= \sum_{k=0}^{n} \binom{n}{k}^2 \underbrace{-\binom{n}{\lceil \frac{n}{2} \rceil}^2}_{\text{if } n \text{ odd}} - \binom{2n}{n-1} \underbrace{+\binom{n}{\lceil \frac{n}{2} \rceil}^2}_{\text{if } n \text{ odd}} \\
&= \binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n} = C_n.
\end{aligned}$$

Alternative 2: We will use Young tableaux but we will not use the formula from part (a) and not the hint. First of all observe that there is a bijection between standard Young tableaux with rows of length at most 2 and those of height at most 2 by transposition (like matrices). We will count the number of tableaux of height at most 2.

Recall from Problem 30 on this sheet that there are C_n standard Young tableaux of shape (n, n). We will describe a bijection between these tableaux and pairs of standard Young tableaux of the same shape of n.

Consider a standard Young tableaux of shape (n, n). Observe that the squares filled with numbers $1, \dots, n$ form a standard Young (sub)tableaux T_1, since all numbers below or right of a number $i > n$ are even larger than i. Moreover taking the squares filled with the numbers $n+1, \dots, 2n$, rotating by π and replacing i with $2n-i+1$, yields a standard

Young tableaux T_2 of height at most 2. By construction T_1 and T_2 have the same shape of height at most 2.

The other way round we obtain a tableaux of shape (n, n) from a pair (T_1, T_2) by rotating T_2 by π, replacing i with $2n - i + 1$ and "gluing" the result to T_1.

This shows that the number of pairs of tableaux of height at most 2 equals C_n. Hence we are done since we argued in part (a) that a_n equals the number of such pairs. $\square$