Solution Sheet 11

Due date: **July 10, 2017, 15:30**. Discussion of solutions: July 10, 2017.

Problem 31. **6 points**

For some positve integer n the boolean lattice of dimension n is a poset B_n whose elements are all subsets of $[n]$ and $S <_{B_n} S'$ if and only if S is a proper subset of S'. Prove that for each poset P there is an integer n such that P is a subposet of B_n.

Solution.

Let n denote the number of elements in P and label the elements of P from x_1 to x_n arbitrarily. Now for some element x_i in P let $D[x_i]$ denote its closed downset, that is, the set of all integers j with $x_j \leq_P x_i$. This establishes an injective map from P to B_n. We claim that $D[x_j] \subseteq D[x_i]$ (that is $D[x_j] \leq_{B_n} D[x_i]$) if and only if $x_j \leq_P x_i$. Observe that $i \in D[x_i]$ for each $i \in [n]$. Hence, if $D[x_j] \subseteq D[x_i]$ then $j \in D[x_i]$ and thus $x_j \leq_P x_i$. The other way round suppose that $x_j \leq_P x_i$. Now for any $j' \in D[x_j]$ we have $x_{j'} \leq_P x_j \leq_P x_i$. Hence $x_{j'} \leq_P x_i$ by transitivity of $\leq_P$ and thus $j' \in D[x_i]$. Altogether $D[x_j] \subseteq D[x_i]$. □

Problem 32. **6 points**

For a poset $P = (X, \leq)$ consider the families $\mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \mathcal{A}_3$ and $\mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \mathcal{C}_3$ given by

$$\mathcal{A}_1 = \{A \subseteq X \mid A \text{ antichain in } P\}, \quad \mathcal{A}_2 = \{A \in \mathcal{A}_1 \mid A \text{ maximal}\}, \quad \mathcal{A}_3 = \{A \in \mathcal{A}_2 \mid A \text{ largest}\},$$
$$\mathcal{C}_1 = \{C \subseteq X \mid C \text{ chain in } P\}, \quad \mathcal{C}_2 = \{C \in \mathcal{C}_1 \mid C \text{ maximal}\}, \quad \mathcal{C}_3 = \{C \in \mathcal{C}_2 \mid C \text{ longest}\}.$$

For each of the following statements determine all ordered pairs (i, j) with $i, j \in \{1, 2, 3\}$ such that the statement holds for all posets P. Justify your answer!

(a) There is some $A \in \mathcal{A}_i$ such that for each $C \in \mathcal{C}_j$ we have $|A \cap C| = 1$.

(b) There is some $C \in \mathcal{C}_j$ such that for each $A \in \mathcal{A}_i$ we have $|A \cap C| = 1$.

(c) For each $A \in \mathcal{A}_i$ and for each $C \in \mathcal{C}_j$ we have $|A \cap C| = 1$.

Solution.

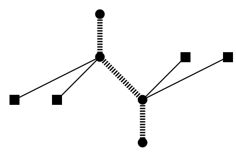
(a) Observe that if the statement holds for some pair (i, j), then it holds also for smaller i and larger j. We claim that the statement holds for all posets P if and only if $i \in \{1, 2\}$ and $j \in \{2, 3\}$.

First, consider the pair $(2, 2)$ and some arbitrary poset P. Let A denote the antichain that consists of all minimal elements of P. Then A is maximal, so $A \in \mathcal{A}_2$. Moreover each chain $C \in \mathcal{C}_2$ is maximal, so its smallest element x is a minimum of P. Thus $|A \cap C| = |\{x\}| = 1$. This shows that the statement holds for the pair $(2, 2)$, and thus holds for all pairs (i, j) with $i \in \{1, 2\}$ and $j \in \{2, 3\}$.

Next, consider the pair $(3, 3)$ and the poset P in the picture below. Let C denote the dashed chain and let A denote the antichain formed by the boxes. Then C is a longest chain in P, so $C \in \mathcal{C}_3$ and A is the unique largest antichain in P. Since $A \cap C = \emptyset$

we have that the statement does not hold for P for the pair $(3,3)$. This shows that the statement does not hold for P for any pair (i,j) with $i=3$ and $j \in \{1,2,3\}$.

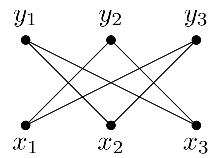
Finally consider the pair $(1,1)$. We see that for each antichain A in P there is some chain C in P with $A \cap C = \emptyset$ (take $C = \{x\}$ for some $x \notin A$). This shows that the statement does not hold for P for the pair $(1,1)$ and hence does not hold for any pair (i,j) with $i \in \{1,2,3\}$ and $j=1$.



(b) Observe that if the statement holds for some pair (i,j), then it holds also for larger i and smaller j. We claim that the statement holds for all posets P if and only if $i=3$ and $j \in \{1,2\}$.

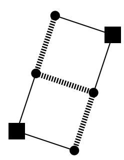
First, consider the pair $(3,2)$ and some arbitrary poset P. Let $A \in \mathcal{A}_3$ be some largest antichain in P. By Dilworth's theorem there is a partition of P into $w(P) = |A|$ many chains. Let C be a chain in such a partition and let C' be any maximal chain in P that contains C. Then $|C' \cap A| = 1$ and $C' \in \mathcal{C}_2$. This shows that the statement holds for the pair $(3,2)$, and thus holds for all pairs (i,j) with $i=3$ and $j \in \{1,2\}$.

Next consider the pair $(2,1)$ and the poset P in the picture below. Each chain in P misses at least one of the antichains $\{x_i, y_i\} \in \mathcal{A}_2$. This shows that the statement does not hold for P and the pair $(2,1)$ and not for any pair (i,j) with $i \in \{1,2\}$ and $j \in \{1,2,3\}$.



Finally consider the pair $(3,3)$ and the poset P' in the picture in part (a). Let C denote the dashed chain in P' and let A denote the antichain formed by the boxes in P'. Similar to part (a) we see that C is the unique longest chain in P' and A is a largest antichain in P'. Since $A \cap C = \emptyset$ we have that the statement does not hold for P' and the pair $(3,3)$.

(c) Observe that if the statement holds for some pair (i,j), then it holds also for larger i and j. Consider the poset P in the picture below. We claim that for P this statement does not hold for any pair (i,j) with $i,\ j \in \{1,2,3\}$. Indeed, the dashed chain C is a longest chain in P, so $C \in \mathcal{C}_3$, and the two black boxes form a largest antichain A in P, so $A \in \mathcal{A}_3$. Since $A \cap C = \emptyset$ the statement does not hold for the pair $(3,3)$ and therefore does not hold for any other pair.



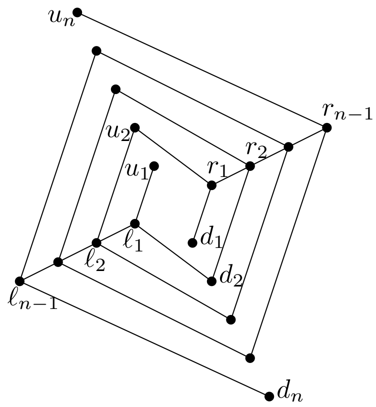
□

Problem 33. **6 points**

For $n \geq 2$ consider the poset $P_n = (X, \leq)$ defined as follows.

$$X = \{u_1, \ldots, u_n\} \cup \{d_1, \ldots, d_n\} \cup \{\ell_1, \ldots, \ell_{n-1}\} \cup \{r_1, \ldots, r_{n-1}\} \text{ with relations}$$

1) $d_i \leq \ell_j \Leftrightarrow i \geq j+1$,	3) $\ell_i \leq u_j \Leftrightarrow i \geq j$,	5) $\ell_i \leq \ell_j \Leftrightarrow i \geq j$,
2) $d_i \leq r_j \Leftrightarrow i \leq j$,	4) $r_i \leq u_j \Leftrightarrow i \leq j-1$,	6) $r_i \leq r_j \Leftrightarrow i \leq j$.



For each $n \geq 2$

(a) characterize all pairs (i, j) with $d_i \leq u_j$ in P_n,

(b) calculate the height of P_n,

(c) calculate the width of P_n,

(d) calculate the dimension of P_n.

Solution.

(a) We claim that we have $d_i \leq u_j$ if and only if $i \neq j$ and $d_i || u_i$ for $i = 1, \ldots, n$.

If $i \leq j-1$, then $d_i \leq r_i \leq u_j$.
If $i \geq j+1$, then $d_i \leq \ell_j \leq u_j$.
If $i = j = 1$, then u_j covers only ℓ_j, but $\ell_j || d_j$. Thus $u_j || d_j$. If $i = j \geq 2$, then u_j covers only ℓ_j and r_{j-1}, but $\ell_j || d_j$ and $r_{j-1} || d_j$. Thus $u_j || d_j$.

(b) We claim that for $n \geq 2$ the height of P_n is exactly $n+1$. Clearly, $h(P_n) \geq n+1$ as $\{d_1, r_1, \ldots, r_{n-1}, u_n\}$ is a chain of size $n+1$ in P_n. Now consider $A_i = \{\ell_{i,r_i}$ for $i \in [n-1]$, $A_n = \{d_1, \ldots, d_n\}$, and $A_{n+1} = \{u_1, \ldots, u_n\}$. Then each set A_i is an antichain for $i \in [n+1]$ and $P_n = \bigcup_{i \in [n+1]} A_i$. Thus $h(P_n) = n+1$.

(c) We claim that for $n \geq 2$ the width of P_n is exactly n. Clearly, $w(P_n) \geq n$ as $\{u_1, \ldots, u_n\}$ is an antichain of size n in P_n.

We shall give a partition of P_n into n chain to show that $w(P_n) \leq n$. Indeed such a partition is given by chains $C_i = \{d_i, r_i, u_{i+1}\}$, $i \in n-1$, and $C_n = \{d_n, \ell_1, \ldots, \ell_{n-1}, u_1\}$.

(d) We claim that $\dim(P_n) = n$. Indeed, we know that $\dim(P_n) \leq w(P) = n$, where $w(P_n) = n$ was shown in part (c). On the other hand, the first item simply says that $\{u_1, \ldots, u_n, d_1, \ldots, d_n\}$ induces the standard example S_n in P_n. Hence we have $\dim(P_n) \geq \dim(S_n) = n$. □